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OPTIMAL CYCLIC SCHEDULING OF MULTISTAGE CONTINUOUS MULTIPRODUCT PLANTS

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Abstract—This paper addresses the problem of optimizing cyclic schedules of multiproduct continuous plants that consist of a sequence of stages each involving one production line that are interconnected by storage tanks. The problem involves a combinatorial part (sequencing of products) and a continuous part (duration of production runs and inventory levels). The problem is formulated as a large scale mixed-integer nonlinear program (MINLP) model that involves nondifferentiabilities in the inventory levels for the storage tanks. Binary variables and mixed-integer constraints are used to remove these nondifferentiabilities. A solution method based on variants of the generalized Benders decomposition and outer approximation is proposed for this scheduling problem. The method consists of an MINLP subproblem in which cycle times and inventory levels are optimized for a fixed sequence, and an MILP master problem that determines the optimal sequence of production. Examples are presented to compare the proposed decomposition method with the direct solution of the MINLP using an augmented penalty version of the outer-approximation method. The results show that the computational requirements can be greatly reduced in problems involving several hundred 0–1 variables, and several thousand continuous variables and constraints.

INTRODUCTION

Extensive reviews in batch processing have been recently reported in the literature (Reklaitis, 1991, 1992; Rippin, 1992). Many of these problems can be posed as mixed integer optimization programs since the corresponding mathematical optimization models involve both discrete and continuous variables that must satisfy a set of equality and inequality constraints (Grossmann *et al.*, 1992). These scheduling and planning problems are in fact often posed as mixed integer linear programming (MILP) models.

While scheduling of batch processes has received considerable attention in the literature, much less work has been reported in the scheduling of continuous multiproduct plants despite their practical importance. These arise frequently in chemical process industries. Petroleum refineries which have to process different crudes, polymer and specialty chemicals plants, and paper machines are three examples. Furthermore, one of the current trends in the chemical industry is to move towards continuous flexible multiproduct plants that can respond more quickly to demand changes and to the processing of a variety of products.

In terms of continuous multiproduct processes, the planning and scheduling problem has been studied in the Operations Research literature with the assumptions of infinite horizon and single stage; this is known as the economic lot-scheduling problem (Elmaghraby, 1978). A recent solution method for solving this problem includes the cutting planes by Magnanti and Vachani (1990). The special case of a two product mix was applied in an oil refinery (Kella, 1991). The two stage problem was studied by Buzacott and Ozkarahan (1983) with variable processing rates but restricted to two products.

In the chemical processing industry, the current available tools for solving the scheduling and planning problems are single and/or multiperiod linear programming problems (Picaseno-Gamiz, 1989). There are a few studies in scheduling and planning reported in chemical engineering. Sahinidis and Grossmann (1991a) considered the cyclic scheduling of continuous multiproduct plants with parallel lines and formulated the problem as a large-scale mixed integer nonlinear programming (MINLP) problem. The authors developed a solution method based on generalized Benders decomposition for which they were able to solve problems with up to 800 binary variables, 23,000 continuous variables and 3000 constraints. While cyclic schedules are typically considered for plants operating at long term horizons with constant demand rates, short term schedules are considered for variable demands. An example of the latter is the planning of multiproduct continuous processes under resource constraints addressed by Kondili *et al.* (1993). Their formulation resulted in a

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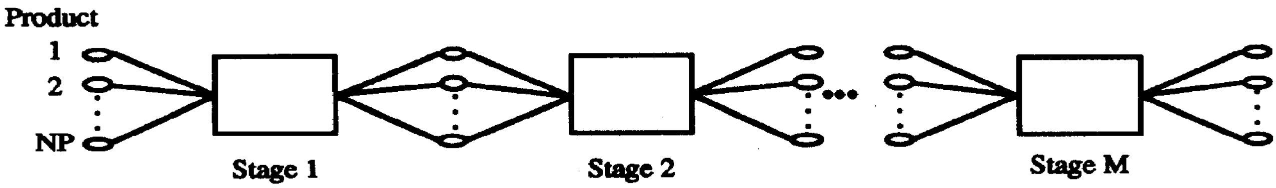

Fig. 1. Multiproduct continuous plant with several stages interconnected by intermediate inventory tanks.

mixed integer linear program that was applied to the short term production of a cement plant.

It is the objective of this paper to propose an optimization model and solution method for the cyclic scheduling of multistage continuous multiproduct plants. A large scale MINLP model is developed that can handle intermediate storage between stages as well as sequence dependent changeovers. It is first shown that the handling of nondifferentiabilities for intermediate storage are best handled with 0–1 variables. An effective decomposition method is then proposed that combines generalized Benders decomposition and the outer-approximation method. Numerical results are presented to illustrate the usefulness of the proposed model and solution method.

PROBLEM DEFINITION

The specific scheduling problem that will be addressed in this work can be stated as follows:

Given is a number of specified products that are to be manufactured in a plant consisting of several stages that are interconnected by intermediate inventory tanks for each product (Fig. 1). It is assumed that every product must be processed in the same sequence throughout all the stages (i.e. flowshop plant). Each stage consists of one production line with equipment which is interconnected with a fixed topology. Transition times that arise between the processing of two successive products are sequence dependent. Given are also constant demand rates in the form of lower bounds that are to be satisfied. The problem then consists in determining the following items for a cyclic schedule:

(A) sequencing of products
(B) length of cycle time
(C) length of production times
(D) amounts of products to be produced
(E) levels of intermediate storage and final product inventories.

The criterion used is the maximization of profit, that includes income from the sales of the products and inventory and transition costs.

MOTIVATING EXAMPLE

In order to provide some insights into the nature of the decisions and trade-offs involved in the scheduling of a continuous multiproduct plant, consider the following example that arises for instance in plants for manufacturing lubricants. A continuous multiproduct plant consisting of 2 stages must be scheduled to produce a minimum of 50 kg/h of A, 100 kg/h of B and 250 kg/h of C in a cyclic operation mode. Each product has to go through the 2 stages in the same order. Data on processing rates, (sequence dependent) transition times, prices of products, intermediate storage costs and final product costs are given in Table 1. The transition costs are $760 from product A to any product in all stages, $750 for B and $770 for C. Several feasible alternative schedules are shown in Fig. 2.

Alternatives 1 and 2 have the same sequence A–B–C as seen in Fig. 2. Alternative 1 has a cycle time of 44.2 h while in 2 its value is 165.4 h. Since in alternative 1 the production is repeated more frequently, the inventory cost is reduced. On the other hand, this implies more frequent transitions. For example, the transition times correspond to 54.3%

Table 1. Data for motivating example

Prices/Demands

Product	Price ($/ton)	Demand (kg/h)
A	150	50
B	400	100
C	650	250

Processing Rates/Inventory Costs

	Stage 1		Stage 2	
Product	Processing rate (kg/h)	Intermediate storage ($/ton)	Processing rate (kg/h)	Final inventory ($/ton.h)
A	800	140.6	900	4.06
B	1200	140.6	600	4.06
C	1000	140.6	1100	4.06

Transition Times (Sequence Dependent) (h)

	Stage 1			Stage 2		
Product	A	B	C	A	B	C
A	—	10	3	–	7	3
B	3	—	6	3	—	10
C	8	3	—	3	0	—

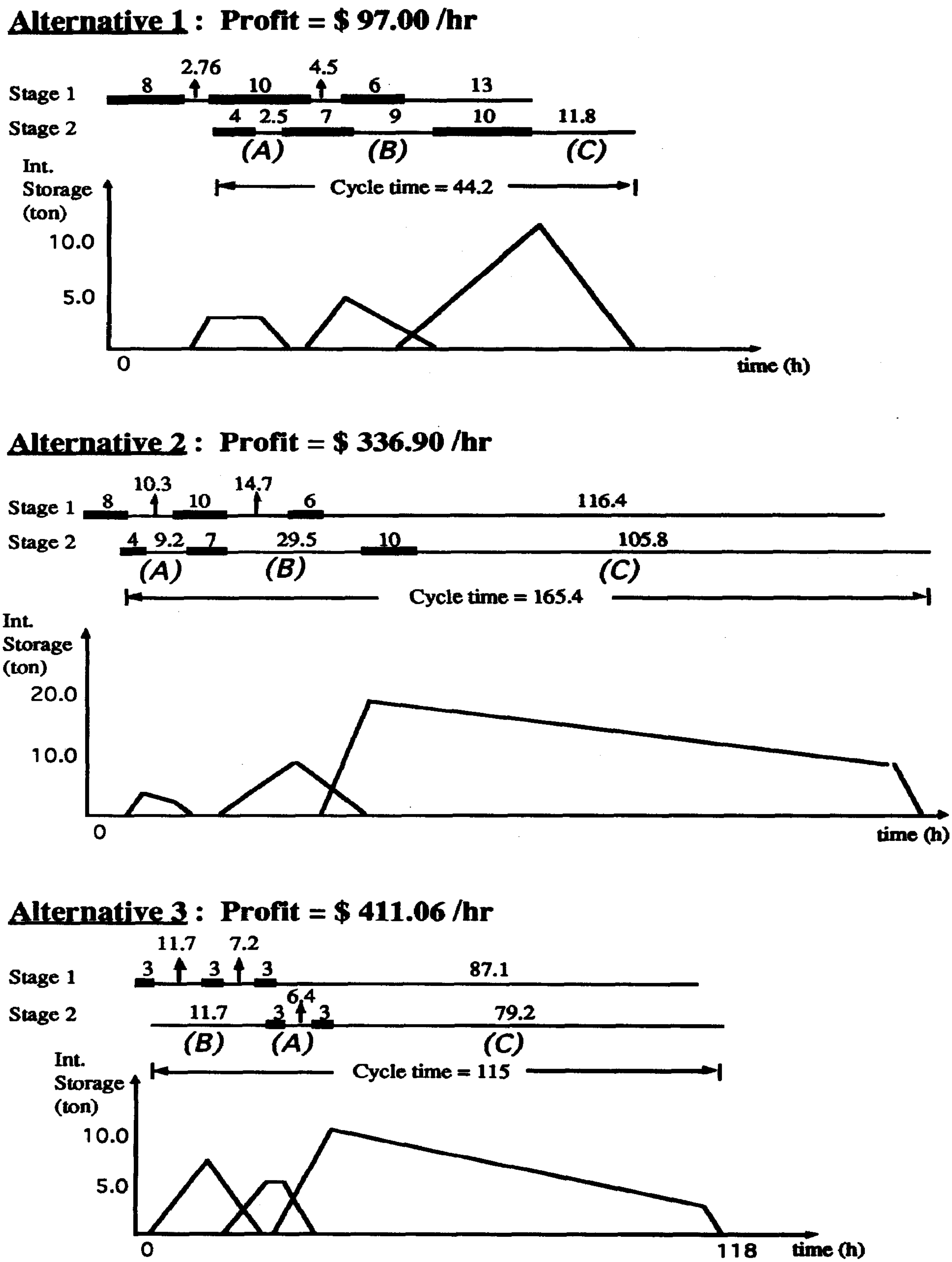

Fig. 2. Feasible schedules for motivating example.

of the total cycle time in alternative 1. A larger profit is attained in alternative 2 due mainly to an increase in productivity of C (see Table 2).

The sequence in alternative 3 is B–A–C. Note from Table 1 that smaller transition times are incurred with this sequence (9 h per cycle vs 24 h per cycle in stage 1, 6 h per cycle vs 21 h per cycle in stage 2). The cycle time is 115 h with which an even larger productivity of C and corresponding increase in profit are obtained (see also Example 2).

One can see in Table 2 that the demand of A (the less valuable product) is satisfied at its minimum for all alternatives and that in alternatives 2 and 3 the production of C is greatly increased. In terms of both intermediate and final product storage there is a direct relation with the cycle time.

What this example shows is that the interactions and trade-offs involved in the scheduling of continuous multiproduct plants are complex. There exists an optimal cycle time that is largely determined by the sequencing of products, inventory levels and transition times. Therefore, there is a clear incentive for developing systematic techniques to handle the scheduling problem.

MATHEMATICAL MODEL FOR SCHEDULING

In order to simplify the scheduling problem the proposed model will rely on the assumption that each product must be processed in the same sequence at each stage. This assumption is in general reasonable since in most practical systems the difference between the production rates in successive stages is not very large. The basic ideas of the model are as follows:

(a) NP time slots are postulated at each stage (Fig. 3), where NP is the number of products. These time slots represent the sequence of the processing of the products. However, the product assignment to each slot and its processing times are variables to be determined. Binary variables are used to model the potential assignment of product to time slots. Transition time variables are activated depending on the assignment.

(b) The inventory cost between stages is assumed to depend on the maximum level of material that is accumulated between stages as in Buzacott and Ozkarahan (1983). Constraints will be developed to determine the maximum inventory levels that ensure that the start time in stage $m+1$ is such that it does not deplete the contents of the inventory from the products processed in stage m. As seen in Fig. 4 the maximum level of inventory depends on the relative start times of two successive stages. The break points in the inventory profiles can be modeled with equations that involve max–min operators which are nondifferentiable [see equation (6) and Appendix A for a more detailed description]. It is important to note that since the process is continuous, the start time of a subsequent stage can begin before the end time has been reached in the previous stage [see Fig. 4(b)].

(c) The inventory cost of final products is assumed to depend on the average amount of material to be stored. This average amount can be determined in closed form as discussed in Sahinidis and Grossmann (1991a) (see also Appendix A).

Table 2. Production rates and inventory levels of final products

Alternative	Production rates (kg/h)			Inventory levels of final products (ton)		
	A	B	C	A	B	C
1	50	122	293	2.09	4.29	9.52
2	50	107	700	7.81	14.52	42.31
3	50	122	757	5.43	11.20	27.17

In the proposed model, the following are the indices:

Products $i, j = 1, \ldots, \mathrm{NP}$
Stages $m = 1, \ldots, M$
Time slots $k = 1, \ldots, \mathrm{NP}$

The following are the variables:

y_{ik}	0–1 variable to denote if product i is assigned to slot k
z_{ijk}	0–1 variable to denote if product i is preceded by product j in slot k
Ts_{km}	start time of stage m in slot k
Te_{km}	end time of stage m in slot k
Tp_{km}	processing time of stage m in slot k
Tpp_{ikm}	processing time of product i at stage m in slot k
Tc	cycle time
Wp_{ikm}	amount produced of product i at stage m in slot k
γ_{km}	processing rate of stage m in slot k
α_{km}	mass balance coefficient in stage m in slot k
$I0_{km}$, $I1_{km}$	break point for inventory level between stages m and $m+1$ (see Fig. 4)
$I2_{km}$, $I3_{km}$	break point for inventory level between stages m and $m+1$ (see Fig. 4)
$I\max_{km}$	maximum inventory level at slot k between stages m and $m+1$
Ip_{ikm}	inventory capacity at slot k between stages m and $m+1$ for product i

and the following are the parameters:

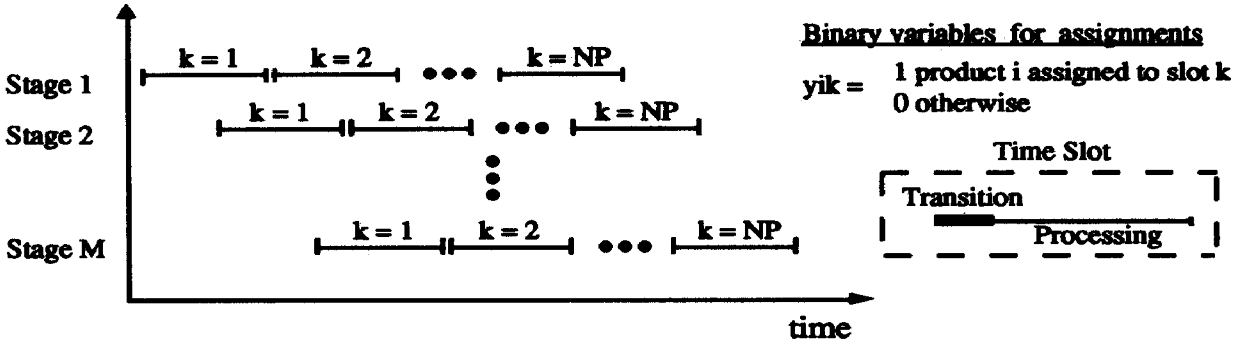

Fig. 3. Time slots of a plant with one production line per stage.

d_i	minimum demand rate of product i
p_i	price of product i
α_{im}	mass balance coefficient from product i in stage m
γp_{im}	processing rate of product i at stage m
τ_{ijm}	transition time from product i to product j in stage m
$Cinv_{im}$	cost coefficient for inventory of product i at stage m
$Cinvf_i$	cost coefficient for inventory of final product i
Ctr_{ij}	transition cost between product i and product j
U^T_{im}	upper hand processing time of product i at stage m
U^I_{im}	upper hand of inventory of product i at stage m.

The MINLP model for the scheduling problem is as follows.

Maximize

$$\text{Profit} = \sum_i \sum_k p_i \frac{Wp_{ikM}}{Tc} - \sum_i \sum_k \sum_m Cinv_{im} \times \frac{Ip_{ikm}}{Tc} - \sum_i \sum_j \sum_k Ctr_{ij} \frac{z_{ijk}}{Tc} - \frac{1}{2} \sum_i \sum_k Cinvf_i \left(\gamma p_{iM} - \frac{Wp_{ikM}}{Tc} \right) Tpp_{ikM} \quad (1)$$

subject to:

$$\sum_k y_{ik} = 1 \ \forall i \quad (2a)$$

$$\sum_i y_{ik} = 1 \quad \forall k \quad (2b)$$

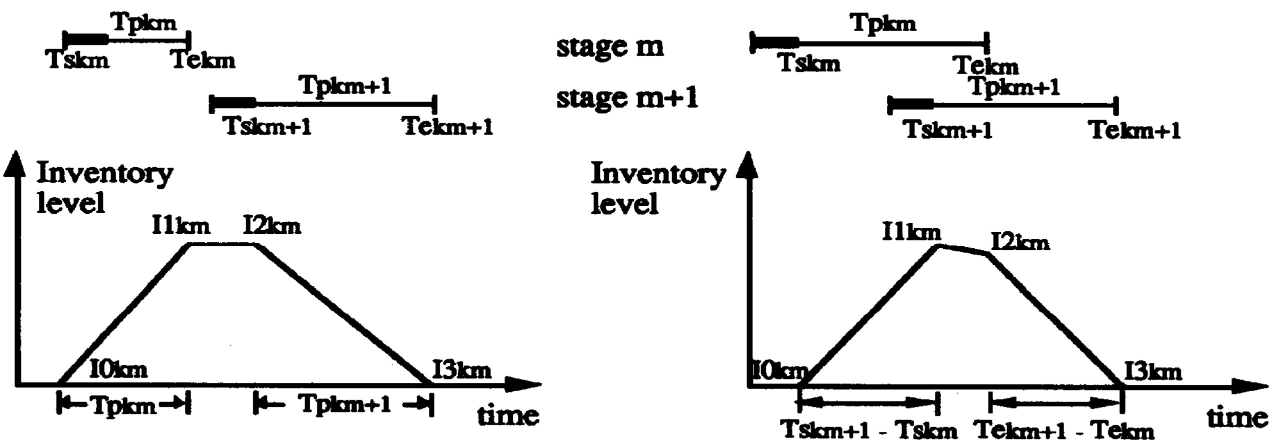

Fig. 4. Inventory levels of products between stages (time slot k).

$$z_{ijk} \geq y_{ik} + y_{jk--1} - 1 \qquad \forall i,j\ \forall k \qquad (3)$$

$$Wp_{ikm} = \gamma p_{im} Tpp_{ikm} \qquad \forall i \quad \forall k \qquad \forall m \qquad (4a)$$

$$Wp_{ikm} = \alpha_{im+1} Wp_{ikm+1} \quad \forall i \quad \forall k \qquad m = 1 \ldots M-1 \qquad (4b)$$

$$\gamma_{km} = \sum_i \gamma p_{im} y_{ik} \qquad \forall k \qquad \forall m \qquad (4c)$$

$$Tpp_{ikm} - U^T_{im} y_{ik} \leq 0 \qquad \forall i \quad \forall k \qquad \forall m \qquad (4d)$$

$$Tp_{km} = \sum_i Tpp_{ikm} \qquad \forall k \qquad \forall m \qquad (4e)$$

$$Tp_{km} = Te_{km} - Ts_{km} \qquad \forall k \qquad \forall m \qquad (5a)$$

$$Ts_{k+1m} = Te_{km} + \sum_i \sum_j \tau_{ijm} z_{ijk+1} \qquad k = 1 \ldots NP-1\ \forall m \qquad (5b)$$

$$Ts_{11} = \sum_i \sum_j \tau_{ij1} z_{ij1} \qquad (5c)$$

$$Ts_{km} \leq Ts_{km+1} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (5d)$$

$$Te_{km} \leq Te_{km+1} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (5e)$$

$$Tc \geq \sum_k \left(Tp_{km} + \sum_i \sum_j \tau_{ijm} z_{ijk} \right) \qquad \forall m \qquad (5f)$$

$$I1_{km} = \gamma_{km} \min\{Ts_{km+1} - Ts_{km}, Tp_{km}\} + I0_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6a)$$

$$I2_{km} = (\gamma_{km} - \alpha_{km+1}\gamma_{km+1}) \max\{0, Te_{km} - Ts_{km+1}\} + I1_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6b)$$

$$I3_{km} = -\alpha_{km+1}\gamma_{km+1} \min\{Te_{km+1} - Te_{km}, Tp_{km+1}\} + I2_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6c)$$

$$0 \leq I1_{km} \leq I\max_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6d)$$

$$0 \leq I2_{km} \leq I\max_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6e)$$

$$0 \leq I3_{km} \leq I\max_{km} \qquad \forall k \qquad m = 1 \ldots M-1 \qquad (6f)$$

$$I\max_{km} = \sum_i Ip_{ikm} \qquad \forall k \qquad \forall m \qquad (6g)$$

$$Ip_{ikm} - U^I_{im} y_{ik} \leq 0 \qquad \forall i \quad \forall k \qquad m = 1 \ldots M \qquad (6h)$$

$$\sum_k Wp_{ikM} \geq d_i T_c \qquad \forall i \qquad (7)$$

$$y_{ik}, z_{ijk} = 0, 1$$

$$Wp_{ikm}, \gamma_{km}, Tpp_{ikm}, Tp_{km}, Te_{km}, Ts_{km}, Tc, I\max_{km}, Ip_{ikm} \geq 0.$$

In (1), profit is defined as the sum of sales revenue, the inventory cost for the intermediate products as a function of maximum levels, the transition cost and the average inventory cost for the final product. The cost coefficients in the objective function are assumed to be scaled so as to yield a profit in annual terms (\$/yr). Also note that the inventory cost coefficients $Cinv_{im}$ and $Cinvf_i$ have different units (\$/ton) and \$/ton.h, respectively.

According to constraints (2) exactly one time slot must be assigned to one product and vice versa. Again, the total number of time slots will be exactly the total number of products.

Equation (3) states the fact that the transition variable:

$$z_{ijk} = \begin{cases} 1 & \text{if product } i \text{ is preceded by product } j \text{ at the beginning of time slot } k, \\ 0 & \text{otherwise} \end{cases}$$

has to be linked with the assignment variables y_{ik} in such a way that $z_{ijk} = 1$ if both y_{ik} and y_{jk--1} are one. On the other hand, if at least one of them is zero, the constraint becomes redundant and, since transitions represent cost terms in the objective function, the optimization of the model will drive these transition variables to zero. It should be noted that "--1" is the cyclic operator and it denotes the previous time slot in the cycle. If, for example, there are three times slots (1, 2 and 3) the notation yields $1--1 = 3$.

Another way of enforcing the condition in (3) is to use the following set of constraints:

$$\sum_i z_{ijk} = y_{jk--1} \ \forall j\ \forall k \qquad (8)$$

$$\sum_j z_{ijk} = yik \quad \forall i\ \forall k. \qquad (9)$$

According to (8), exactly one transition from product j occurs in the beginning of any time slot if and only if j was being produced during the previous time slot. In (9), exactly one transition to product i occurs in the time slot if and only if i is being produced during that time slot. These constraints were proposed in Sahinidis and Grossmann (1991a). The authors showed that this formulation is tighter than the constraints in (3). Furthermore, it requires fewer constraints.

In equations (4) the mass balances, processing rates and amounts produced are considered. The mass balance between stages m and $m+1$ is established for all products, according to the mass balance coefficients in (4b). These account for the fact that material can be added or removed at each stage m. In (4a) one calculates the amounts produced in all stages and for all products which are proportional to the production time. The proportionality constant is the processing rate (which is product and stage dependent). In (4c), the processing rate in time slot k is defined according to the assignment variable y_{ik}. Equation (4d) states that the time devoted to the production of product i at time slot k and stage m is zero unless product i is assigned to that time slot (in this case $y_{ik}=1$). The parameter U^{T}_{im} is a valid upper bound for the processing times which are related in (4e). Note that according to (2) and (4d) exactly one processing time is non zero in the summation term of (4e).

Equations (5) represent the timing constraints. In (5a) the length of the processing time is defined as the difference between the end time and start time. In order to calculate the start time $Ts_{k+1,m}$ [equations (5b) and (5c)] one has to account for the end time in the previous time slot (Te_{km}) and any transition time τ_{ijm} (which are sequence and stage dependent). In stage 1 the start time of the first time slot depends only on the transition times, since no previous time slot exists. Constraints (5d) and (5e) ensure that production in every stage always initiates and terminates after production in the previous stage. The length of the cycle time is defined in (5f) as the maximum over all stages m. The length of the cycle time in stage m is equal to the summation of the lengths of all the time slots (which include processing and transition times).

Inventory levels for intermediates (see Appendix A) are represented through the constraints in (6) which are nondifferentiable since they involve min and max operators to define inventory levels as functions of end and start times which may have different relative positions. The break points in the inventory profiles $I1_{km}$, $I2_{km}$ and $I3_{km}$ are modeled in (6a), (6b) and (6c) (see Fig. 4). These values are bounded in (6d), (6e) and (6f) by the variable $I\max_{km}$ which determines the maximum inventory level for the intermediate products (Ip_{ikm}) in (6g) and (6h). The parameter U^{I}_{im} is an upper bound for the inventory levels. Again, according to (2) and (6h) only one term is non zero.

Constraint (7) states that demand must be satisfied for all products in the plant and that production may be exceeded. It should be noted that in these problems it is not always easy to specify feasible demands. A simple procedure is presented in Appendix B to test the feasibility of specified demands.

The model given by equations (1)–(9) corresponds to a nondifferentiable MINLP problem. Note that the only nonlinearities present in the model are the objective function (due to the cycle time Tc and the last term) and constraints (6a), (6b) and (6c) since the processing rates γ_{km} for each slot k and stage m are variables.

It is also important to note that the model can be applied to the particular case in which the sequence of products is fixed (y_{ik} is known). Constraints (2) can then be removed. The variables z_{ijk} are directly calculated as well as the processing rates γ_{km}. In this case the only nonlinear term is the objective function.

EXACT LINEARIZATION TECHNIQUE

The nonlinearities in constraints (6a), (6b) and (6c) involve bilinear terms in which one of the variables is the processing rate γ_{km}. In order to remove these nonlinearities we first disaggregate the variables denoting the start and end times of the time slots. The variables Tsp_{ikm} and Te_{ikm} are introduced, where the former is the start time of product i in stage m and slot k and the latter is the end time of product i in stage m and slot k. The aggregated values Ts_{km} and Te_{km} are equal to the summation of the disaggregated times over all the products. Equations (10b) and (11b) are required in order to guarantee that, together with (2) only one value of the disaggregated variables is non-zero. Note also that with this disaggregation equation (4c) is not required since the variables γ_{km} can be expressed in terms of the parameters γp_{im}. This yields the following constraints:

$$Ts_{km}=\sum_i Tsp_{ikm} \qquad \forall k\ m=1\ldots M-1 \quad (10a)$$

$$Tsp_{ikm}-U^{\mathrm{T}}_{im}y_{ik}\leqslant 0 \quad \forall i\ \forall k\ m=1\ldots M-1 \quad (10b)$$

$$Te_{km}=\sum_i Tep_{ikm} \qquad \forall k\ m=1\ldots M-1 \quad (11a)$$

$$Tep_{ikm}-U^{\mathrm{T}}_{im}y_{ik}\leqslant 0 \quad \forall i\ \forall k\ m=1\ldots M-1 \quad (11b)$$

$$I1_{km}=\sum_i (\gamma p_{im}\min\{Tsp_{ikm+1}-Tsp_{ikm},\ Tpp_{ikm}\})+I0_{km}$$
$$\forall k\ m=1\ldots M-1 \quad (12a)$$

$$I2_{km}=\sum_i [(\gamma p_{im}-\alpha_{im+1}\gamma p_{im+1})$$
$$\max\{0,\ Tep_{ikm}-Tsp_{ikm+1}\}]+I1_{km}$$
$$\forall k\ m=1\ldots M-1 \quad (12b)$$

$$I3_{km}=-\sum_i (\alpha_{im+1}\gamma p_{im+1}$$
$$\min\{Tep_{ikm+1}-Tep_{ikm},\ Tpp_{ikm+1}\})+I2_{km}$$
$$\forall k\ m=1\ldots M-1. \quad (12c)$$

One final constraint should be added to the inventory levels. Since the scheduling is for cyclic operation, one must ensure that there will be no build-up or depletion after a cycle. One possible way is to define as in equation (13):

$$I0_{km} = I3_{km} \qquad \forall k\ m = 1 \ldots M-1. \quad (13)$$

With these reformulations the optimization problem becomes linearly constrained, although nondifferentiabilities in (12) are present.

TREATMENT OF NONDIFFERENTIABILITIES

The nondifferentiabilities can be removed either by using smooth approximations (Duran and Grossmann, 1986b; Balakrishna and Biegler, 1992) or by modeling the max–min operators with 0–1 binary variables (Raman and Grossmann, 1991).

Smooth approximation

The smoothing technique proposed by Balakrishna and Biegler (1992) will be considered. It is based on a hyperbolic approximation to convert the nondifferentiable function $\phi = \max\{0, f(x)\}$ to a continuous nonlinear function of the form:

$$\phi \approx \frac{\sqrt{f(x)^2+\varepsilon^2}}{2} + \frac{f(x)}{2} \quad (14)$$

note that:

$$\text{if } f(x) \geq 0 \text{ (for } \varepsilon \to 0\text{)}: \phi \approx f(x)$$
$$\text{if } f(x) < 0 \text{ (for } \varepsilon \to 0\text{)}: \phi \approx 0$$

Typical values for ε are in the range 0.01–0.0001. Note also that in this reformulation even if $f(x)$ is a linear function the approximation will introduce nonlinearities to the model. Hence the reformulation can be applied directly to the nonlinear equations in (6b). For equations (6a) and (6c) before applying the approximation, one has to rewrite the min operators as follows:

$$\min\{f1(x), f2(x)\} = -\max\{-f1(x), -f2(x)\} = f1(x) - \max\{0, f1(x) - f2(x)\}. \quad (15)$$

The reformulated equations are then:

$$I1_{km} = I0_{km} + \gamma_{km}\{Tp_{km} - \tfrac{1}{2}[Tp_{km} - \Delta Ts_{km+1} + \sqrt{(Tp_{km} - \Delta Ts_{km+1})^2 + \varepsilon^2}]\} \quad \forall k\ m = 1 \ldots M-1 \quad (16a)$$

where $\Delta Ts_{km+1} = Ts_{km+1} - Ts_{km}$

$$I2_{km} = I1_{km} + (\gamma_{km} - \alpha_{km+1}\gamma_{km+1})\{\tfrac{1}{2}[Te_{km} - Ts_{km+1} + \sqrt{(Te_{km} - Ts_{km+1})^2 + \varepsilon^2}]\} \quad \forall k\ m = 1 \ldots M-1 \quad (16b)$$

$$I3_{km} = I2_{km} - \alpha_{km+1}\gamma_{km+1}\{Tp_{km+1} - \tfrac{1}{2}[Tp_{km+1} - \Delta Te_{km+1} + \sqrt{(Tp_{km} - \Delta Te_{km+1})^2 + \varepsilon^2}]\} \quad \forall k\ m = 1 \ldots M-1 \quad (16c)$$

where $\Delta Te_{km+1} = Te_{km+1} - Te_{km}$.

The main difficulty with this procedure, is that apart from yielding nonlinear functions it involves nonconvexities. In fact, as will be shown later in the paper, computational experience has revealed the existence of multiple local optima even for small problems with the use of the equations in (16) (see Example 1).

Mixed integer representation

The function $\phi = \max\{0, f(x)\}$ can be represented by the two following inequalities as discussed in Raman and Grossmann (1991):

$$\begin{aligned} &0 \leq \phi - f(x) \leq U_1(1-y) \\ &0 \leq \phi \leq U_2 y. \end{aligned} \quad (17)$$

In the above formulation y is a 0–1 binary variable and $U_i (i = 1, 2)$ are upper bounds. Note also that:

$$y = 0 \rightarrow \phi = 0$$
$$y = 1 \rightarrow \phi = f(x).$$

Applying the same transformation to the min operators [see equation (15)], equations (12a), (12b) and (12c) can be written as:

$$\left.\begin{aligned} &I1_{km} = \sum_i [\gamma_{im}(Tpp_{ikm} - \phi^1_{ikm})] + I0_{km} \\ &0 \leq \phi^1_{ikm} - Tpp_{ikm} + Tsp_{ikm+1} - Tsp_{ikm} \leq U^1_{ikm}(1 - x^1_{ikm}) \\ &0 \leq \phi^1_{ikm} \leq U^1_{ikm} x^1_{ikm} \end{aligned}\right\} \forall k\ m = 1 \ldots M-1 \quad (18a)$$

$$\left.\begin{aligned} &I2_{km} = \sum_i [(\gamma_{im} - \alpha_{km+1}\gamma_{im+1})\phi^2_{ikm}] + I1_{km} \\ &0 \leq \phi^2_{ikm} - Tep_{ikm} + Tsp_{ikm+1} \leq U^2_{ikm}(1 - x^2_{ikm}) \\ &0 \leq \phi^2_{ikm} \leq U^2_{ikm} x^2_{ikm} \end{aligned}\right\} \forall k\ m = 1 \ldots M-1 \quad (18b)$$

$$\left.\begin{aligned} &I3_{km} = -\sum_i [\alpha_{km+1}\gamma_{im+1} (Tpp_{ikm+1} - \phi^3_{ikm})] + I2_{km} \\ &0 \leq \phi^3_{ikm} - Tpp_{ikm+1} + Tep_{ikm+1} - Tep_{ikm} \leq U^3_{ikm}(1 - x^3_{ikm}) \\ &0 \leq \phi^3_{ikm} \leq U^3_{ikm} x^3_{ikm} \end{aligned}\right\} \forall k\ m = 1 \ldots M-1 \quad (18c)$$

where x^1_{ikm}, x^2_{ikm} and x^3_{ikm} are 0–1 binary variables.

Although the equations remain linear, one can clearly see that the mixed integer representation

Table 3. Data for Example 1

Product	Price ($/ton)	Demand (kg/h)
A	800	100
B	150	50
C	1100	730

	Stage 1		Stage 2	
Product	Proc. rate (kg/h)	Inventory cost ($/ton)	Proc. rate (kg/h)	Inventory cost ($/ton.h)
A	1200	50	600	4.06
B	800	50	900	4.06
C	1000	50	1100	4.06

introduces a large number of binary variables. In fact, the number of binary variables added to the problem is $NP^2 \times (M-1)$ (NP is the number of products and M is the number of stages).

In summary, the proposed scheduling model with the mixed integer representation is as follows:

$$\begin{array}{lll} \text{maximize objective function (1)} & & \\ \text{subject to constraints} & (2), (8), (9), (4a\text{–}b) & \\ & (4d\text{–}e), (5), (10), & (P1) \\ & (11), (18), (13), & \\ & (6d\text{–}h), (7). & \end{array}$$

The use of the smooth approximation introduces equations (16) in place of constraints (10), (11) and (18).

The following example illustrates the performance of both the smooth approximations and mixed integer representation for modeling the non-differentiabilities.

EXAMPLE 1—FIXED SEQUENCE PROBLEM

The example considered is a small problem involving the production of three products (A, B and C) in two stages. Consider the case of the fixed sequence B–A–C. The problem reduces to a NLP with the use of smooth approximation and to the MINLP (P1) for the mixed integer representation. Note that for the last case the only nonlinear equation is the objective function. The data for the problem are given in Table 3. The transition costs are the same as for the motivating example.

Given an arbitrary initial point the MINLP converges to the solution shown with the Gantt chart in Fig. 5(a). The optimal profit of $723/h is obtained with a cycle time of 207 h. Using the NLP model with the smooth approximation, the solution converges to two different optima, depending on the choice of the initial point. Using as the initial point the solution given by the MINLP case, the NLP converges to the same solution. However, given the same arbitrary point used as the initial guess for the MINLP, the optimal profit obtained is only $687/h and the corresponding cycle time is 201 h [see Fig. 5(b)].

It is important to note that both solutions are feasible. This means that the demands are satisfied. For example, in the first solution the production rates for A, B and C are 100, 50 and 738 kg/h and for the second case are 100, 50 and 735 kg/h respectively. The main difference between the two solutions is in the arrangement of the intermediate inventory levels. One can see in Fig. 5 that apart from the fact of having a smaller profit, the second solution is clearly worse since the inventory levels are much larger. Besides, there is no apparent justification for having the shift in the schedule of stage 2 with respect to stage 1.

The MINLP model was composed of 9 binary 0–1 variables, 143 continuous variables and 164 constraints; 10.1 CPU (IBM-6000) were required to solve this problem with DICOPT++ (Viswanathan and Grossmann, 1990). The NLP model with the smooth approximation involved 134 variables and 137 constraints; 2.1 CPU s were required with GAMS/MINOS (Brooke *et al.*, 1988).

The results obtained so far for fixed sequences have not revealed existence of multiple optima for the mixed integer representation, although no guarantee of global optimality (convexity characterizations) have been derived for the model. On the other hand, a large number of binary variables is introduced to the model which can make the solution computationally expensive.

SOLUTION PROCEDURE

As shown in the last section the max–min operators in the inventory constraints (12) can be removed using the differentiable MINLP formulation (P1) which can be solved in principle by the methods developed in the literature: generalized Benders decomposition (Geoffrion, 1972; Sahinidis and Grossmann, 1991b) and outer approximation (Duran and Grossmann, 1986a; Kocis and Grossmann, 1989; Viswanathan and Grossmann,

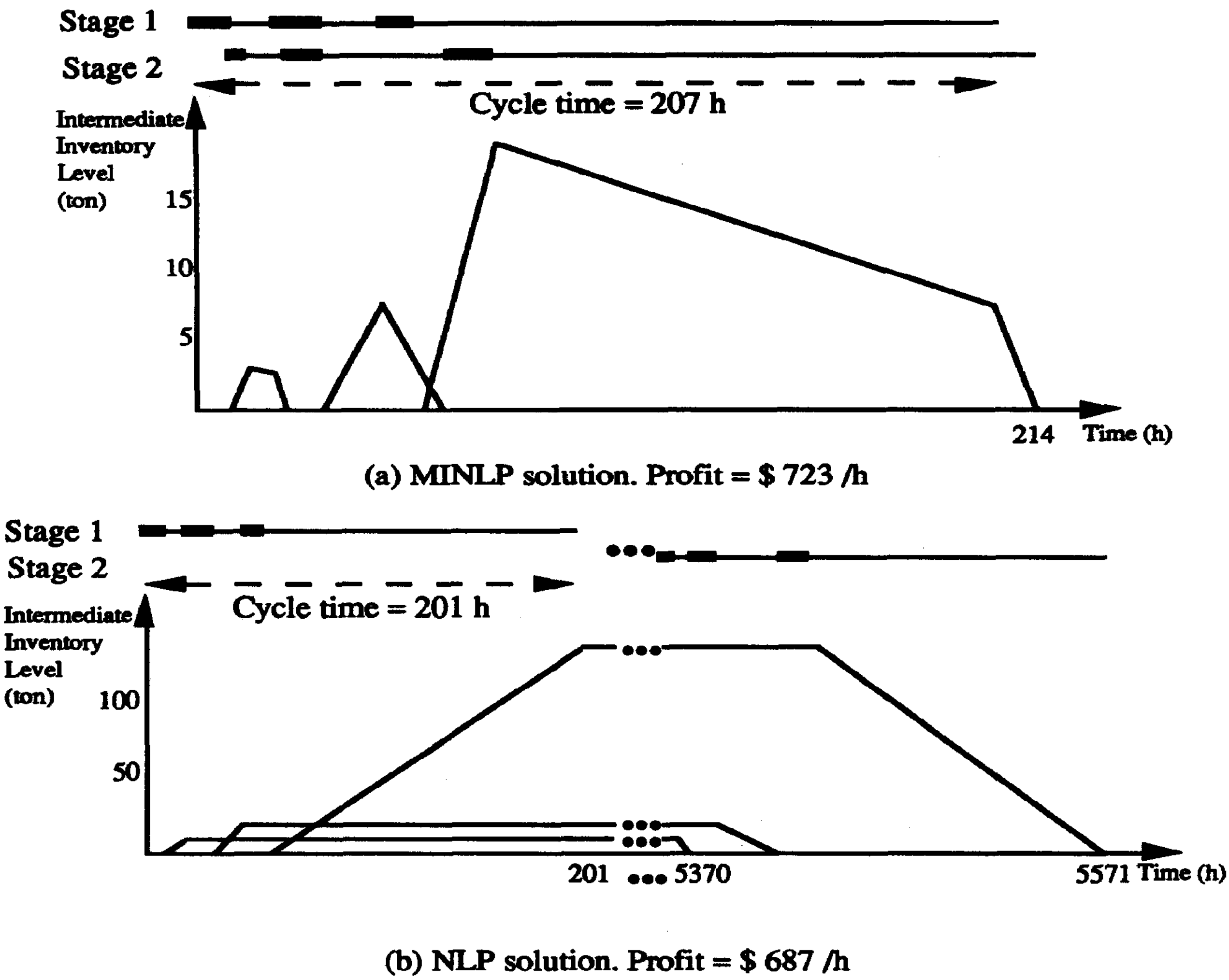

Fig. 5. Gantt chart for the solutions of Example 1.

1990). However, even with these reformulations it is clear that both size and complexity are major issues in the solution of such a model.

An algorithm which is based on combining generalized Benders decomposition [v-GBD method as defined by Sahinidis and Grossmann (1991b)] and outer approximation (augmented penalty method) is proposed in this paper and compared to the outer approximation method implemented in the code DICOPT++ (Viswanathan and Grossmann, 1990). Figure 6 illustrates the proposed solution method. The motivation behind this method is to consider the optimization of the cycle time for a fixed sequence as a subproblem and the optimization of the sequence as part of a master problem. The basic steps are as follows:

Step 1 (initialization)

Determine an initial assignment by fixing the binary variables y_{ik}. Set $\text{Profit}^U = \Sigma_i p_i y_{iM}$, $\text{Profit}^L = -\infty$ and $q = 1$. Select the convergence parameter ε, i.e. the desired ε% optimality of the solution.

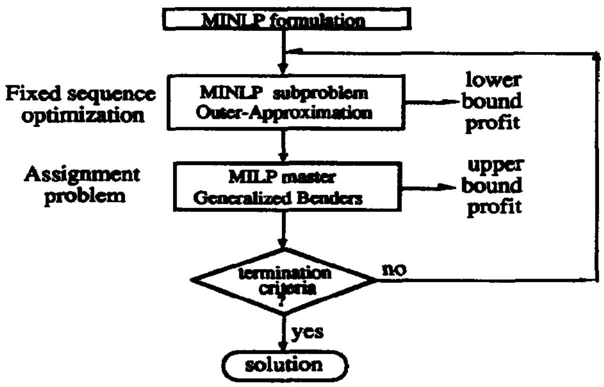

Fig. 6. Proposed solution method.

Step 2

Solve the MINLP subproblem (P1) with fixed y_{ik} using DICOPT++ to optimize the cycle time. The solution of this problem Profit^q can be used to update the lower bound: $\text{Profit}^L = \max\{\text{Profit}^L, \text{Profit}^q\}$. If $(\text{Profit}^U - \text{Profit}^L)/\text{Profit}^U \leq \varepsilon/100$, stop. Otherwise, go to Step 3.

Step 3

Construct and solve a Benders MILP master problem to determine new assignment variables y_{ik}. The solution of this problem yields an upper bound for the profit. The MILP master is given by:

$$\text{Profit}^U = \max \eta \tag{19}$$

subject to:

$$\eta \leq L^r(y_{ik}) \qquad r = 1 \ldots q \tag{20}$$

$$\sum_k y_{ik} = 1 \qquad \forall i \tag{2a}$$

$$\sum_i y_{ik} = 1 \qquad \forall k \tag{2b}$$

$$\eta \in R^1 \quad y_{ik} = \{0, 1\} \quad \forall i \ \forall k$$

where L^r is the Lagrangian defined as:

$$\begin{aligned} L^r = \text{Profit}^r &+ \sum_i \sum_k \sum_m \mu^1_{ikm}(Tpp^r_{ikm} - U^T_{im} y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^2_{ikm}(Ip^r_{ikm} - U^I_{im} y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^3_{ikm}(x^{1r}_{ikm} - y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^4_{ikm}(x^{2r}_{ikm} - y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^5_{ikm}(x^{3r}_{ikm} - y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^6_{ikm}(Tsp^r_{ikm} - U^T_{im} y_{ik}) \\ &+ \sum_i \sum_k \sum_{m=1}^{M-1} \mu^7_{ikm}(Tep^r_{ikm} - U^T_{im} y_{ik}). \end{aligned} \tag{21}$$

In the above expression Profit^r is the value of the objective function in (1) and $\mu^1_{ikm}, \mu^2_{ikm}, \mu^6_{ikm}, \mu^7_{ikm}$ are the Lagrange multipliers of constraints (4d), (6h), (10b), (11b) respectively. The Lagrange multipliers $\mu^3_{ikm}, \mu^4_{ikm}, \mu^5_{ikm}$ correspond to constraints (22a), (22b) and (22c), which will be described later in this section.

Step 4

If $(\text{Profit}^U - \text{Profit}^L)/\text{Profit}^U \leq \varepsilon/100$, stop. Otherwise, set $q = q + 1$ and return to step 2.

Note that in step 2 the interesting feature is that an MINLP subproblem is being solved as opposed to an NLP subproblem. The justification for this is simply that the solution to the MINLP in step 2 corresponds to the solution of an equivalent NLP which, however, is nondifferentiable. This subproblem is equivalent to the scheduling problem for fixed sequence. The binary variables are x^1_{ikm}, x^2_{ikm} and x^2_{ikm} which are used to model the nondifferentiabilities in (18). In order to reduce the search in the subproblem, the following logic constraints can be added:

$$y_{ik} - x^1_{ikm} \geq 0 \quad \forall i \forall k \ m = 1 \ldots M-1 \tag{22a}$$

$$y_{ik} - x^2_{ikm} \geq 0 \quad \forall i \forall k \ m = 1 \ldots M-1 \tag{22b}$$

$$y_{ik} - x^3_{ikm} \geq 0 \quad \forall i \forall k \ m = 1 \ldots M-1. \tag{22c}$$

The cuts (22a), (22b), (22c) state that when the product i is not assigned to time slot k, i.e. $y_{ik} = 0$, all the corresponding binary variables will also have the value zero. Furthermore, the following additional cuts can be introduced:

$$\sum_k x^1_{ikm} \leq 1 \quad \forall i \quad m = 1 \ldots M-1 \tag{23a}$$

$$\sum_k x^2_{ikm} \leq 1 \quad \forall i \quad m = 1 \ldots M-1 \tag{23b}$$

$$\sum_k x^3_{ikm} \leq 1 \quad \forall i \quad m = 1 \ldots M-1 \tag{23c}$$

$$\sum_i x^1_{ikm} \leq 1 \quad \forall k \quad m = 1 \ldots M-1 \tag{24a}$$

$$\sum_i x^2_{ikm} \leq 1 \quad \forall k \quad m = 1 \ldots M-1 \tag{24b}$$

$$\sum_i x^3_{ikm} \leq 1 \quad \forall k \quad m = 1 \ldots M-1. \tag{24c}$$

Constraints (23a), (23b), (23c) and (24a), (24b), (24c) reflect the fact that exactly one product will be assigned to one time slot and vice versa. Therefore at most one binary variable will be necessary to describe the inventories for every product and every time slot.

It should also be noted that in order to avoid infeasible MINLP subproblems it is possible to add slack variables to the demand constraints in (7) as discussed in Sahinidis and Grossmann (1991a).

As a means of achieving faster convergence, the master problem of Benders in step 3 is strengthened by the incorporation of valid upper bounds. Such bounds are linearizations of the subproblem objective function, resulting in an outer approximation strategy (Duran and Grossmann, 1986a). Instead of linearizing the objective function itself, the technique is performed on a valid upper bound function,

$$\overline{\text{Profit}} = \sum_i \sum_k p_i \frac{Wp_{ikm}}{Tc} - \frac{C_1}{Tc} - \frac{1}{2} \sum_i \sum_k \text{Cinv}f_i \left(\gamma p_{iM} - \frac{Wp_{ikM}}{Tc} \right) \times Tpp_{ikM}. \tag{25}$$

In (25) C_1 is an underestimation of the transition costs over the stages. This is taken as the summation of the smallest possible transition costs for the products. Based on the solution $x^q = (Wp^q_{ikm}, Tc^q, Tpp^q_{ikm})$ of the subproblem in iteration q we can include the following outerlinearization of (25) into the master problem of iteration $q+1$:

$$\eta \geq \text{Profit} \geq \overline{\text{Profit}}(x) \geq \overline{\text{Profit}}(x^q) + \nabla\overline{\text{Profit}}(x^q)(x - x^q). \tag{26}$$

In this way the variables Wp_{ikm}, Tc and Tpp_{ikm} must be included in the master problem along with the constraints (4a), (4b), (4d), (4e) and (5f).

REMARKS

The proposed decomposition method may fail to identify the global optimum. This is due to the presence of nondifferentiabilities in the subproblems which, even if they are modeled with mixed integer constraints, makes the projected (master) problem nonconvex. Therefore, the Benders cuts are not guaranteed to provide valid bounds to the objective function (see Sahinidis and Grossmann, 1991b). Appendix C illustrates through a small example one situation where the Benders method [v-GBD method as defined by Sahinidis and Grossmann (1991b)] might fail to find the global optimum on a nondifferentiable MINLP.

It should also be noted that if the MINLP model is solved directly without the decomposition scheme there is in fact no difficulty with the nondifferentiabilities as these are treated with 0–1 variables. In this case the only source of nonconvexity is the objective function which consists of fractional terms [see equation (1)].

COMPUTATIONAL RESULTS

The modeling system GAMS (Brooke *et al.*, 1988) was used in order to implement the scheduling model and its solution method. The solution of the master MILP problems was obtained with the code OSL (IBM, 1991) that performs a branch and bound search. The MINLP problems were solved with the outer approximation code DICOPT++ version 2.4.1 (Viswanathan and Grossmann, 1990), including the subproblem in the proposed method. Three examples are presented: Example 2 illustrates the case in which the selection of a sequence with non-minimal changeover leads to an optimal schedule. In Example 3, a 5 product, 2 stage problem is studied and variations in parameters are performed. Example 4 deals with a larger problem consisting of 8 products in 3 stages.

Table 4. Transition times for Example 2

	Stage 1			Stage 2		
Product	A	B	C	A	B	C
A	—	5	8	—	11	1
B	10	—	3	2	—	5
C	4	7	—	6	1	—

EXAMPLE 2—TRADE-OFF BETWEEN CHANGEOVER TIMES AND TOTAL COST

In principle, a possible way of solving the sequencing and scheduling problem would be to solve the problem in two levels: (1) the sequence is selected first by minimizing the total changeovers; and (2) the optimal schedule is determined for that fixed sequence. The purpose of this example is to demonstrate that this does not necessarily yield the optimal solution. There are complex interactions in the model between transition, inventory costs and profit that do not always allow this problem decomposition.

Consider the case of three products A, B and C being processed in two stages. The prices and demands of the products, processing rates and inventory costs are the same as in the motivating example (see Table 1). The transition times are given in Table 4 with transition costs being proportional to them. The optimal schedule is given in Fig. 7 with sequence A–B–C and total changeover of 34 h (12 for stage 1 and 22 for stage 2). The corresponding profit is \$297/h with a cycle time of 195.6 h.

It is important to note that the optimal schedule does not give rise to the smallest changeover time. The sequence A–C–B has a total changeover of 29 h (25 for stage 1 and 4 for stage 2) but then the optimal profit for this sequence is only \$251/h with a cycle time of 232.6 h.

Despite the results of the example above, it is likely that in many cases decomposing the problem

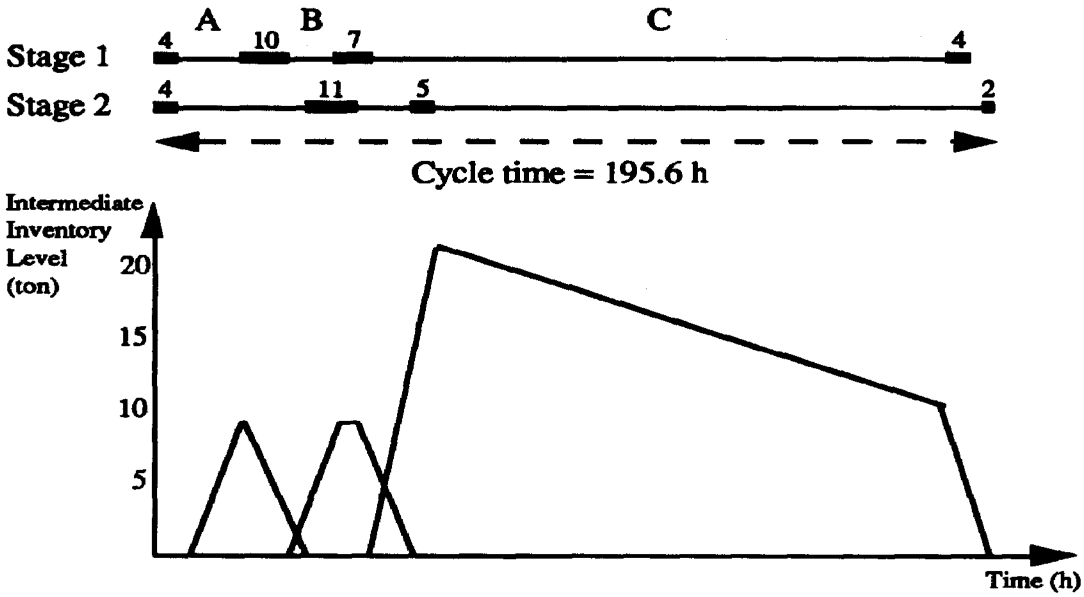

Fig. 7. Optimal solution of Example 2.

by determining first the sequence with minimum changeover times followed by the optimization for fixed sequence may produce the same solution of the overall MINLP model. In this case, however, the solution of the MINLP of step 2 in the proposed procedure is still required.

EXAMPLE 3—A 5 PRODUCT, 2 STAGE PROBLEM

This example consists of 5 products to be processed in 2 stages. Data are shown in Table 5. There are 100 binary variables (25 assignment variables and 75 logical variables for the inventory levels), 527

Table 5. Prices, demands, processing rates, inventory costs and transition times in Example 3

Prices/Demands			
Product	Price ($/ton)	Demand (kg/h)	Demand (%)
A	4000	60	8.0
B	1500	54	7.2
C	6500	500	66.8
D	3000	45	6.0
E	2500	90	12.0

Processing Rates/Inventory Costs	Stage 1		Stage 2	
Product	Processing rate (kg/h)	Intermediate storage ($/ton)	Processing rate (kg/h)	Final inventory ($/ton.h)
A	1170	121.8	1120	4.06
B	1340	121.8	1290	4.06
C	1340	121.8	1340	4.06
D	1210	121.8	1160	4.06
E	1340	121.8	1290	4.06

Transition Times (Sequence Dependent) (h)	Transition times (h)—stage 1					Transition times (h)—stage 2				
Product	A	B	C	D	E	A	B	C	D	E
A	—	10	10	10	10	—	3	3	12	4
B	10	—	1	4	2	3	—	1	12	4
C	10	1	—	4	2	3	1	—	12	4
D	10	4	4	—	4	12	12	12	—	12
E	10	2	2	4	—	4	4	4	12	—

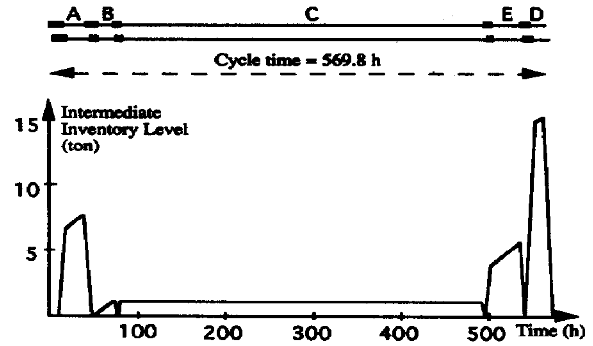

Fig. 8. Gantt chart and intermediate inventory level for Example 3.

Table 6. Production rates and inventory levels of final products for Example 3

Product	Production rates (kg/h)	Increase over minimum demand (kg/h)	Inventory levels of final products (ton)
A	60	0	33.2
B	54	0	29.9
C	989	489	548.1
D	45	0	24.9
E	90	0	49.9

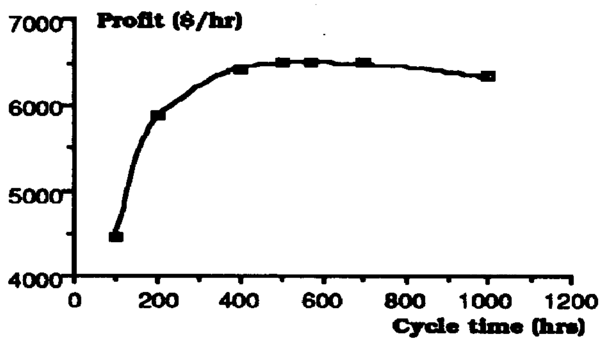

Fig. 9. Profit vs cycle time for Example 3.

continuous variables and 727 equations. The computational times are shown in Table 10. The transition costs are proportional to the transition times in the first stage and each transition hour costs $760/h.

The schedule of the optimal solution, obtained by both the outer approximation and the proposed method is shown in Fig. 8. In the optimal solution the production of C tends to be high (see Table 6), since it is the most valuable product. Also the production of C in the 2 stages is almost simultaneous in order to reduce intermediate inventory levels. The optimal profit is $6513/h for a cycle time of approx. 24 days (569.8 h).

The influence of the cycle time on the profit is illustrated in Fig. 9. We can notice that the profit decreases rapidly for small cycle times since in that case the transition costs are significant. On the other hand, for larger cycle times, as inventory costs are increased, changes in the profit become less sensitive to the selection of the cycle time.

The transition times were modified in order to determine their impact in the schedule (see Fig. 10). Smaller transition times allow reduced cycle time lengths while yielding higher productivity and therefore larger profits are achieved. Also, note that the cycle time and profit are quite sensitive for ± 10% changes in the transition times. The final inventory costs are related to the length of the cycle time in the opposite way: large cost values require a small cycle time, as it is seen in Fig. 11. Changes in the intermediate inventory costs for this specific example did have almost no consequence in the optimal schedule and profit.

EXAMPLE 4—AN 8 PRODUCT, 3 STAGE PROBLEM

A larger system consisting of 8 products and 3 stages is considered. Data for the problem are given in Table 7. The processing rates for each product are different for different stages and the transition costs are all sequence dependent. The demands are also given as a percentage of the total demand of

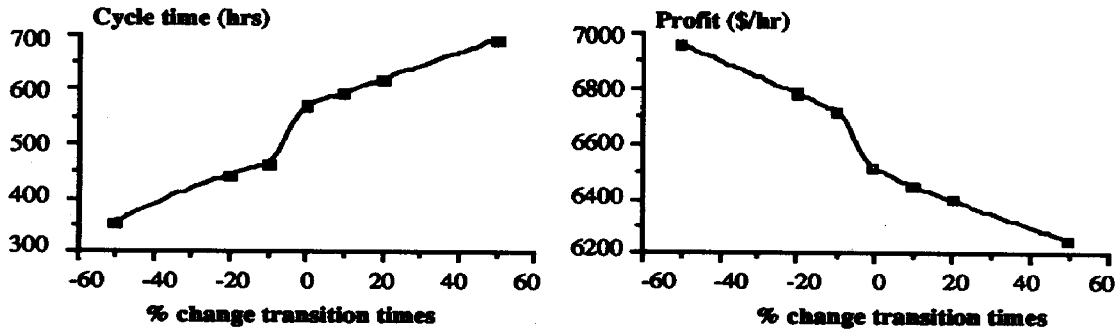

Fig. 10. Optimal cycle time and profit vs variation in transition times (Example 3).

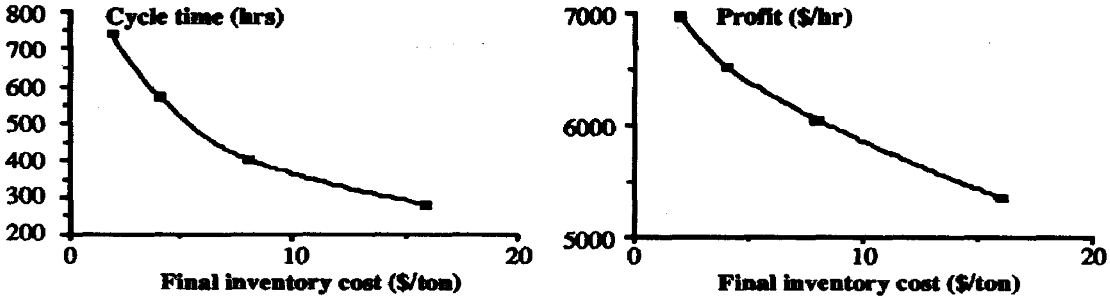

Fig. 11. Optimal cycle time and profit vs final inventory cost (Example 3).

Table 7. Prices, demands, processing rates, inventory costs and transition times in Example 4

Product	Price ($/ton)	Demand (kg/h)	Demand (%)
	Prices/Demands		
A	4000	24	4.4
B	1500	12	2.2
C	6500	350	64.4
D	3000	30	5.5
E	2500	60	11.0
F	3700	10	1.8
G	1000	40	7.4
H	2000	18	3.3

Processing Rates/Inventory Costs

	Stage 1		Stage 2		Stage 3	
Product	Processing rate (kg/h)	Intermediate storage ($/ton)	Processing rate (kg/h)	Intermediate storage ($/ton)	Processing rate (kg/h)	Final inventory ($/ton.h)
A	1170	121.8	1120	121.8	1120	4.06
B	1340	121.8	1290	121.8	1860	4.06
C	1340	121.8	1340	121.8	1340	4.06
D	1210	121.8	1160	121.8	1420	4.06
E	1340	121.8	1290	121.8	1960	4.06
F	1340	121.8	1340	121.8	1340	4.06
G	950	121.8	950	121.8	950	4.06
H	1210	121.8	1290	121.8	1860	4.06

Transition Times (Sequence Dependent) (h)

Product	A	B	C	D	E	F	G	H
A	—	10	10	10	10	10	10	10
B	10	—	1	4	2	2	12	2
C	10	1	—	4	2	2	12	2
D	10	4	4	—	4	4	12	4
E	10	2	2	4	—	1	12	1
F	10	2	2	4	1	—	12	1
G	10	12	12	12	12	12	—	12
H	10	2	2	4	1	1	12	—
				Stage 2				
A	—	3	3	12	4	4	2	3
B	3	—	1	12	4	4	3	2
C	3	1	—	12	4	4	3	2
D	12	12	12	—	12	12	12	12
E	4	4	4	12	—	1	4	4
F	4	4	4	12	1	—	4	4
G	2	3	3	12	4	4	—	3
H	3	2	2	12	4	4	3	—
				Stage 3				
A	—	2	2	8	10	3	10	4
B	2	—	1	8	10	3	10	4
C	2	1	—	8	10	3	10	4
D	8	8	8	—	10	8	10	4
E	10	10	10	10	—	10	8	10
F	3	3	3	8	10	—	10	4
G	10	10	10	10	8	10	—	10
H	4	4	4	8	10	4	10	—

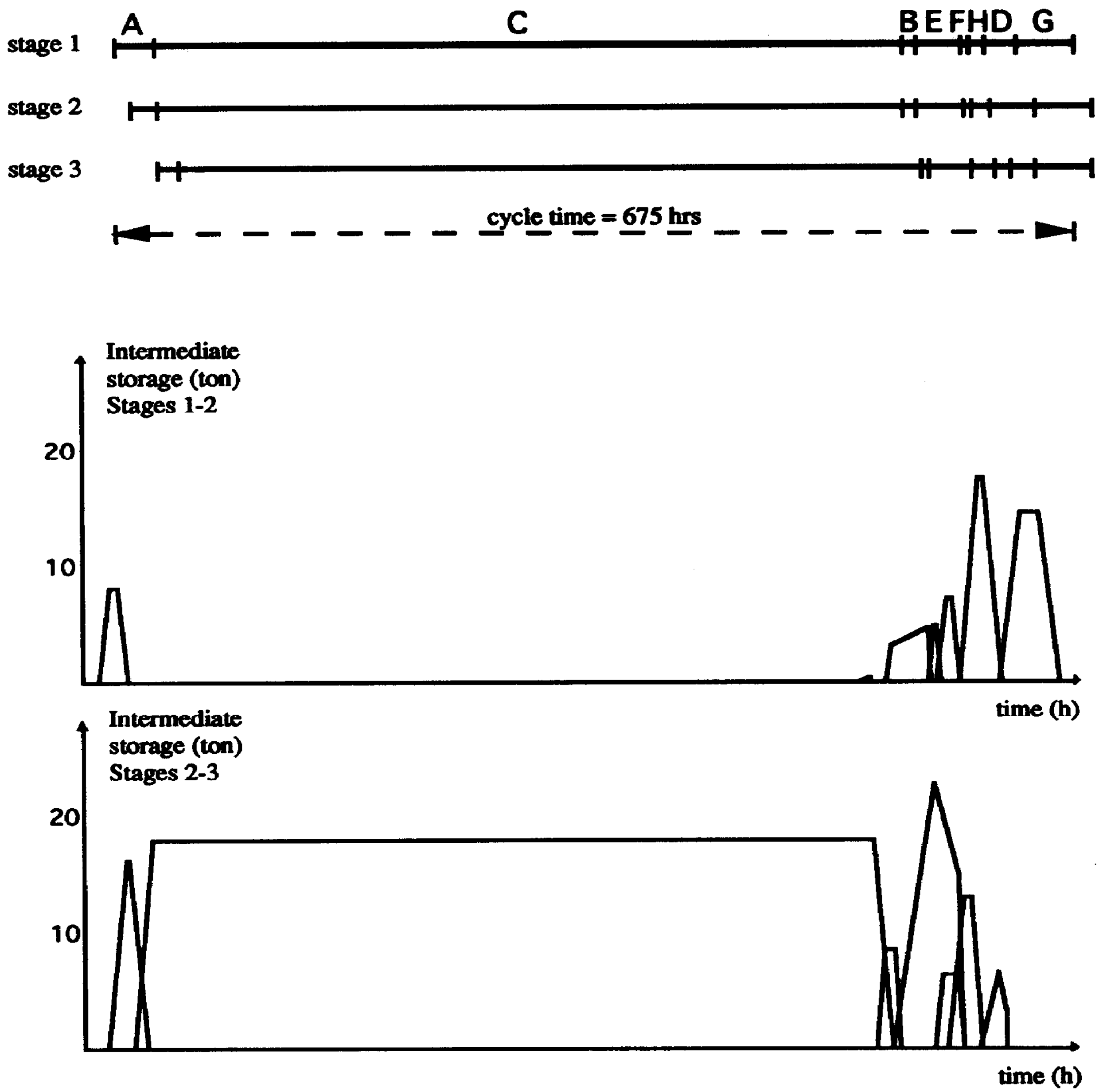

Fig. 12. Optimal schedule for Example 4.

Table 8. Production rates and inventory levels of final products for Example 4

Product	Production rates (kg/h)	Increase over minimum demand (kg/h)	Inventory levels of final products (ton)
A	24	0	16.2
B	12	0	8.1
C	1039.2	689.2	701
D	30	0	20.2
E	60	0	40.5
F	10	0	6.7
G	40	0	27
H	18	0	12.1

544 kg/h. The transition costs are \$750 from product B to all products in the 3 stages, \$770 for C and \$760 for the remaining products. The MINLP problem consists of 448 binary 0–1 variables, 1970 continuous variables and 3002 constraints.

The optimal schedule has a profit of \$6609/h and a cycle time of approx. 28 days (674.6 h). The sequence, processing times and intermediate storage profiles are shown in Fig. 12. Note that the demands are satisfied for all products, but since C is the most valuable product its processing largely exceeds the demand as seen in Table 8. The processing times

Table 9. Processing times and transition times for optimal schedule of Example 4

Stage	A	C	B	E	FF	H	D	G
	Processing Times (Transition Times) (h)							
1	13.8 (10)	523.2 (1)	6.0 (2)	30.2 (1)	5.0 (1)	10.0 (4)	16.7 (12)	28.4 (10)
2	14.5 (3)	523.2 (1)	6.3 (4)	31.41 (1)	5.0 (4)	9.4 (12)	17.4 (12)	28.4 (2)
3	14.5 (2)	523.2 (1)	4.4 (10)	21.8 (10)	5.0 (4)	6.5 (8)	14.3 (10)	28.4 (10)

Table 10. Computational performance of problems

Problems					Solver times (%)	
Products	Stages	Algorithm	Major iterations	CPU time (s)*	Subproblem	MILP master
3	2	Proposed method	2	6.5	94.80	5.20
		DICOPT++	3	3.9	55.98	44.02
4	2	Proposed method	4	16.3	89.66	10.34
		DICOPT++	3	8.7	48.40	51.60
4	4	Proposed method	3	52.5	96.86	3.14
		DICOPT++	3	47.4	29.93	70.07
5	2	Proposed method	4	28.4	84.83	15.17
		DICOPT++	5	44.5	42.11	57.89
5	4	Proposed method	6	252.7	94.44	5.56
		DICOPT++	4	262.6	15.49	84.51
6	3	Proposed method	5	184.1	80.34	19.66
		DICOPT++	5	824.4	9.31	90.69
7	2	Proposed method	4	186.4	33.31	66.69
		DICOPT++	3	319.7	20.21	79.79
7	3	Proposed method	6	707.2	44.02	55.98
		DICOPT++	3	832.3	15.33	84.67
8	3	Proposed method	7	2931.6	21.63	78.37
		DICOPT++	3	5630.9	3.60	96.40
9	2	Proposed method	5	2433.2	9.45	90.55
		DICOPT++	5	3238.6	7.95	92.05

* Workstation HP 9000–750.

and the transition times in all stages for the optimal schedule are shown in Table 9.

COMPUTATIONAL TRENDS

While computational trends with problem size are difficult to establish for MINLP scheduling problems, the computational results of 10 problems for different number of products and stages is reported in Table 10. The corresponding sizes of the MINLP problems are given in Table 11. Note that Examples 2, 3 and 4 presented previously correspond to the first, fourth and ninth entries in these tables.

Table 11. Size of MINLP problems

Problems				
Products	Stages	0–1 variables	Continuous variables	Constraints
3	2	36	191	293
4	2	64	334	486
4	4	160	614	1142
5	2	100	527	727
5	4	250	947	1697
6	3	252	1070	1688
7	2	196	1087	1353
7	3	343	1479	2242
8	3	448	1970	3002
9	2	324	1919	2171

As can be seen from Table 10 the proposed method generally requires, as one might expect, a larger number of major iterations. Note also that for both methods the percentage of CPU time spent on solving the MILP master problem increases with problem size, although the increase is less dramatic in the proposed method due to the fact that the combinatorial part of the model is decomposed. Note also that in the first three problems DICOPT++ requires less time than the proposed method (up to a factor of 2). This trend, however, is reversed in the last five problems where the proposed method typically achieved savings of a factor of 2 and up to 4 in one case. While these trends may not be totally conclusive, they do seem to indicate that the proposed decomposition method becomes increasingly attractive for larger problems.

CONCLUSIONS

The cyclic scheduling of multistage multiproduct continuous plants has been discussed in this paper. It has been shown that the problem can be modeled as a large scale MINLP model which involves non-differentiabilties in the inventory equations. Two alternative ways of handling the nondifferentiabilities were tested: smooth approximation and mixed integer representation. The latter was found to avoid the multiple local optima which were obtained with the former. However, this representation has the drawback of increasing the combinatorial part of the model.

The solution of the scheduling problem can in principle be accomplished with the outer approximation method as implemented in DICOPT++. However, in order to tackle larger problems a solution approach based on generalized Benders decomposition and outer approximation has been proposed. As has been shown with the numerical results, the computational requirements of the proposed method are reasonable. Also, the examples have shown the economic potential and trade-offs involved in the optimization of these systems.

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APPENDIX A

On the Inventory Equations

The inventory equations for intermediates given in (6) are derived, in addition to the modeling of the inventory for final products in equation (1). Although not present in the model, inventory costs for feedstocks can be handled in a similar way as the inventories for final products in the objective function.

Inventories of intermediates

Let i be the product assigned to time slot k. Consider its production between stages m and $m+1$. The intermediate storage increases with production in stage m and decreases with production in stage $m+1$.

$I0_{km}$ is the intermediate inventory level at the beginning of the cycle. With the start of production in stage m its level increases with a constant rate γ_{km} until the beginning of production in stage $m+1$. Another possibility that may happen is that the production in stage $m+1$ begins only after the end of processing in stage m. In this case, the rate of increase in the intermediate level is constant at γ_{km} throughout the processing in stage m. These two possible cases are represented in equation (6a):

$$I1_{km}=\gamma_{km}\min\{Ts_{km+1}-Ts_{km},\ Tp_{km}\}+I0_{km} \qquad \forall k\ m=1\ldots M-1. \quad (6a)$$

At the start of production in stage $m+1$ the inventory level is $I1_{km}$. As seen in equation (6b):

$$I2_{km}=(\gamma_{km}-\alpha_{km+1}\gamma_{km+1})\max\{0,\ Te_{km}-Ts_{km+1}\}+I1_{km} \qquad \forall k\ m=1\ldots M-1 \quad (6b)$$

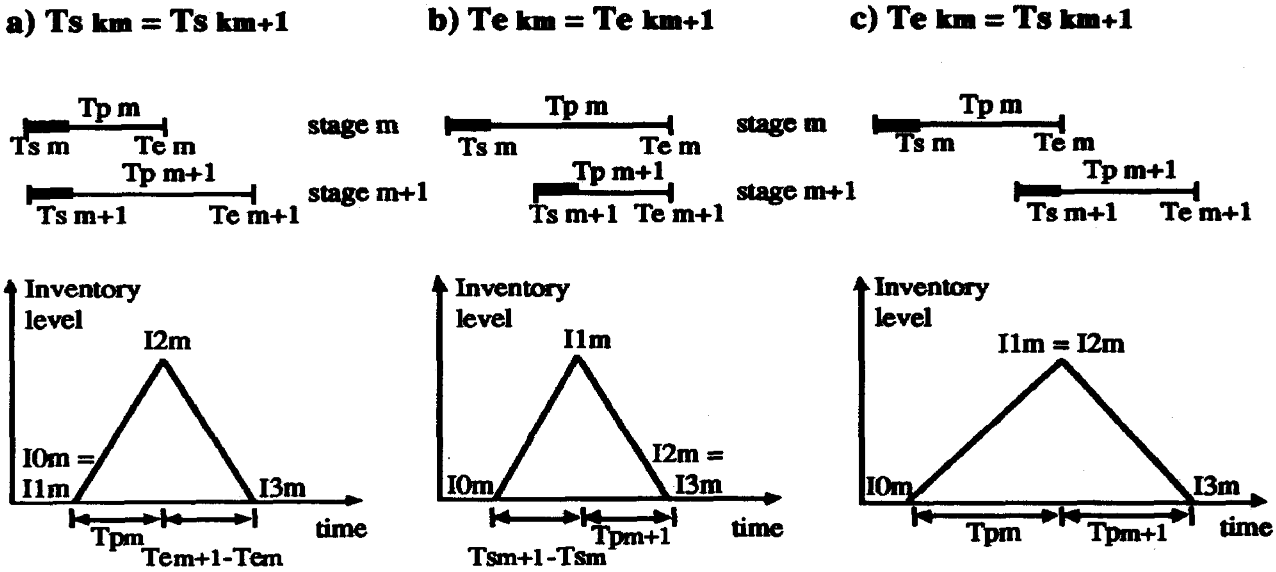

Fig. A1. Intermediate inventory levels for particular cases in time slot k (subscript k omitted for simplicity).

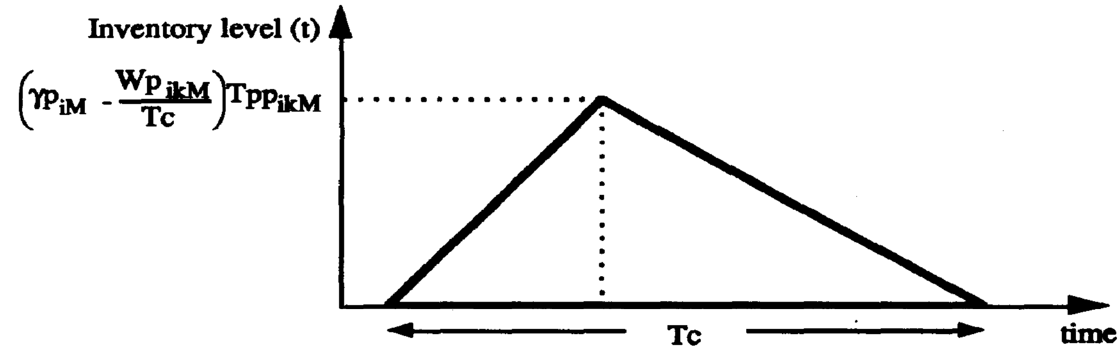

Fig. A2. Final inventory of product i in stage M time slot k.

there is a net production or depletion of the intermediate that depends on the processing rates in both stages. In other words, if $\gamma_{km} > \alpha_{km+1}\gamma_{km+1}$ there is an increase in the intermediate level; if the opposite happens the level decreases. Note that α_{km+1} is the mass balance coefficient in stage $m+1$. In case the production in stage m ends before the start of production in stage $m+1$ the level of intermediate remains constant.

At the intermediate level $I2_{km}$, there is a decrease to the level $I3_{km}$ due to the production in stage $m+1$. Its rate is given by the processing rate γ_{km+1}. The situation is analogous to the one in equation (6a). The production in stage $m+1$ may begin either after the end of production in stage $m+1$ in which case the depletion occurs throughout the processing time Tp_{km+1} or before its end in which case only the difference of the end times has to be accounted for. Equation (6c) represents the two possibilities.

$$I3_{km} = -\alpha_{km+1}\gamma_{km+1}\min\{Te_{km+1} - Te_{km},\, Tp_{km+1}\} + I2_{km} \qquad \forall k\ m = 1 \ldots M-1. \quad (6c)$$

The general cases are represented in Fig. 4. In Fig. 4(a) the processing in stage $m+1$ begins after the production is over, while in Fig. 4(b) it is shown the case in which the processing in stage $m+1$ begins before the end of production in stage m. Note that in the latter case the inventory level decreases between $I1_{km}$ and $I2_{km}(\gamma_{km} < \alpha_{km+1}\gamma_{km+1})$.

Finally, some particular cases may occur. Firstly, the two stages may begin and finish processing at the same time. In this case the processing rates are the same and there is no formation of intermediate inventory. In other words, the two stages behave as a single stage. Secondly, the processing may begin at the same time but due to a smaller processing rate in stage $m+1$ the processing finishes later [see Fig. A1(a)]. Thirdly, the processing in both stages may finish at the same time, but due to a smaller processing rate at stage m its processing starts earlier [Fig. A1(b)]. Finally, it may happen that the processing in stage m finishes exactly at the beginning of processing in stage $m+1$, in which case the inventory profile is as in Fig. A1(c).

Inventories of final products

The inventory cost of the final product is proportional to the integral of the inventory function along time. Given for product i the minimum constant demand rate d_i, the quantity Wp_{ikm}/Tc is the actual demand being satisfied, according to equation (7). Therefore, the average amount produced is given by:

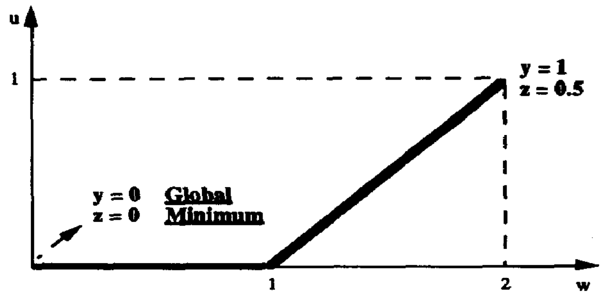

Fig. C1. Feasible region for small example.

$$\dot{W}_i = \sum_k \left(\gamma p_{iM} - \frac{Wp_{ikM}}{Tc}\right) Tpp_{ikM}. \quad \text{(A1)}$$

The corresponding integral can be calculated by the triangle area, as seen in Fig. A2. This area is equal to:

$$\frac{1}{2} Tc\, \dot{W}_i = \frac{1}{2} \sum_k \left(\gamma p_{iM} - \frac{Wp_{ikM}}{Tc}\right) Tpp_{ikM} Tc. \quad \text{(A2)}$$

The resulting cost formula for the final inventory is the following:

$$\text{final inventory cost} = \frac{1}{2} \sum_i \sum_k \left(\gamma p_{iM} - \frac{Wp_{ikM}}{Tc}\right) Tpp_{ikM}. \quad \text{(A3)}$$

Note that in (A3) the cycle time vanished since the profit is calculated over a unit time basis.

APPENDIX B

Feasibility of the Scheduling Problem

A very important issue in the scheduling problem considered in this paper is whether there will be a feasible solution for a given set of demands. Elmaghraby (1978) considers this question for the single machine problem. He proposed an IP formulation that examines the sufficient and necessary conditions. In the present case, this formulation is more complex. However, a simple necessary condition can be derived to check feasibility of a given set of parameters. This condition can be verified prior to the solution of the model.

Consider equation (5f). Dividing both sides by Tc and neglecting the transition times (second term):

$$\sum_k \frac{Tp_{km}}{Tc} \leq 1 \qquad \forall m. \quad \text{(B1)}$$

Using equations (4a) and (4e), (B1) becomes:

$$\sum_i \sum_k \frac{Wp_{ikm}}{Tc} \frac{1}{\gamma p_{im}} \leq 1 \qquad \forall m. \quad \text{(B2)}$$

In order to satisfy the demand rates the amount W_{ikm} of product i in the last stage divided by the cycle time must be equal to the demand rate d_i. Therefore:

$$\sum_k Wp_{ikM} = d_i Tc \qquad \forall i. \quad \text{(B3)}$$

Note that only one term in the summation is non zero. Moreover, equation (B3) is equivalent to equation (7) satisfied at equality. Substituting (B3) in (B2) for $m = M$ yields:

$$\sum_i \frac{d_i}{\gamma p_{iM}} \leq 1. \quad \text{(B4)}$$

For the remaining stages ($m = 1, \ldots, M-1$) the same condition holds except that the mass balance coefficients α_{im} have to be taken into account:

$$\sum_i \frac{d_i}{\Pi_{n=m}^{M-1} \alpha_{in} \gamma p_{im}} \leq 1 \qquad m = 1, \ldots, M-1. \quad \text{(B5)}$$

Note that the inequalities in (B4) and (B5) can be easily tested to verify whether a feasible schedule might exist for a given set of demands.

APPENDIX C

Counterexample for the Nondifferentiable MINLP

Consider the small example below:

$$\text{Min } z = -4u + 2w + \tfrac{1}{2}y \quad \text{(C1)}$$

subject to:

$$u = \max\{0, w-1\} \quad \text{(C2)}$$

$$w \leq 2y \quad \text{(C3)}$$

$$w \geq 2y \quad \text{(C4)}$$

$$u \geq 0 \qquad y = 0, 1. \quad \text{(C5)}$$

The feasible space of this nondifferentiable MILP in the u–w space is shown in Fig. C1. Due to the presence of the max operator in constraint (C2), the feasible region is nonconvex.

The problem possesses only 2 possible solutions at $(w, y, u) = (0, 0, 0)$ and $(w, y, u) = (2, 1, 1)$ with objective function values of $z = 0$ and $z = 1/2$, respectively. Although the first point corresponds to the global minimum, the application of the proposed method may lead to the first point.

Let us consider $y = 1$ as being the initial value of the complicating variable (step 1). The subproblem is solved, yielding the values $(w, y, u) = (2, 1, 1)$. Note that equations (C3) and (C4) are equivalent to $w = 2y$. The Lagrange multipliers for equations (C3) and (C4) are $\lambda_1 = 0$ and $\lambda_2 = -2$. Therefore, the Lagrangian for the master problem is given by:

$$\mathcal{L}(y) = -\tfrac{7}{2}y + 4. \quad \text{(C6)}$$

The solution of the master problem is $\mathcal{L}(1) = 1/2$. Hence, the global minimum at $y = 0$ is not identified.