Enterprise-wide Optimization: Strategies for Integration, Uncertainty, and Decomposition

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Objectives Module

1. **Learn about two major issues in Enterprise-wide Optimization (EWO):**
   Integration and Uncertainty

2. **Learn how to model EWO problems**
   Mathematical Programming Framework

3. **Learn about solution methods for:**
   Stochastic Programming
   Bi-criterion Optimization
   Lagrangean decomposition

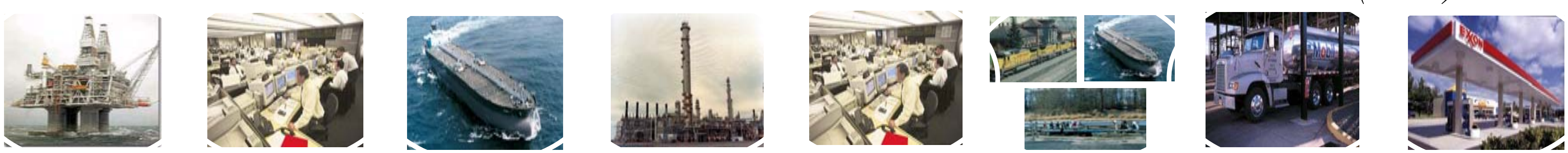
For Background see following sites:
Mixed-integer programming: http://cepac.cheme.cmu.edu/pasilectures/grossmann.htm
Supply Chain Optimization: http://cepac.cheme.cmu.edu/pasilectures/pinto.htm
Enterprise-wide Optimization: http://egon.cheme.cmu.edu/ewocp/slides_seminars.html

Enterprise-wide Optimization (EWO)

EWO involves optimizing the operations of R&D, material supply, manufacturing, distribution of a company to <u>reduce costs and inventories</u>, and to <u>maximize profits, asset utilization, responsiveness</u> .

Petroleum industry

Dennis Houston (2003)



Wellhead | Trading | Transfer of Crude | Refinery Optimization | Schedule Products | Transfer of Products | Terminal Loading | Pump

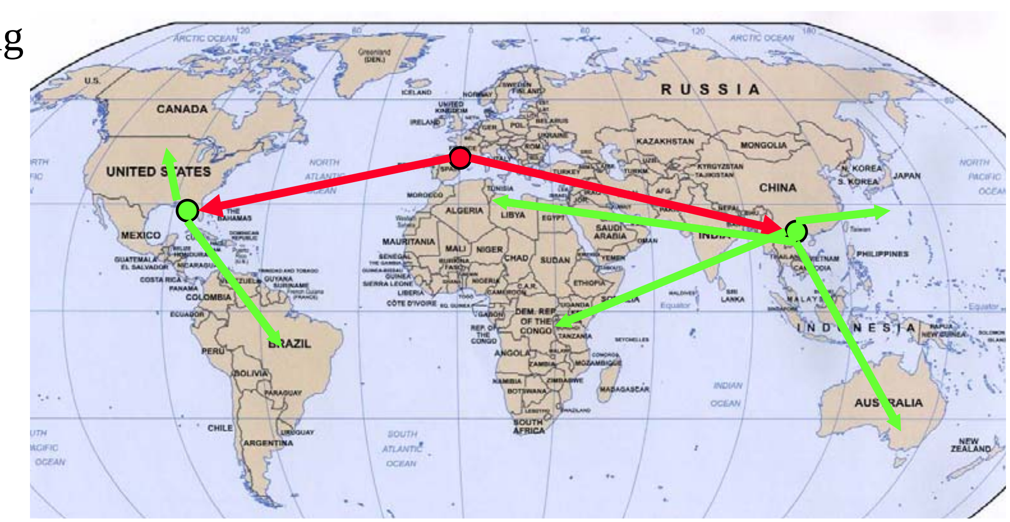
- **The supply chain is large, complex, and highly dynamic**
- **Optimization can have very large financial payout**

Pharmaceutical supply chain

Discovery — ***R&D Pharmaceutical industry*** *(Gardner et al , 2003)* — **Market**

Targets Hits Leads Candidate	Pre-clinical Development	Phase 1	Phase 2a/b	Phase 3	Submission& Approval	Lifecycle Management
2-5 yrs	0.5 - 2 yrs	1 - 2 yrs	1.5 - 3.5 yrs	2.5 - 4 yrs	0.5-2 yrs	10-20 yrs

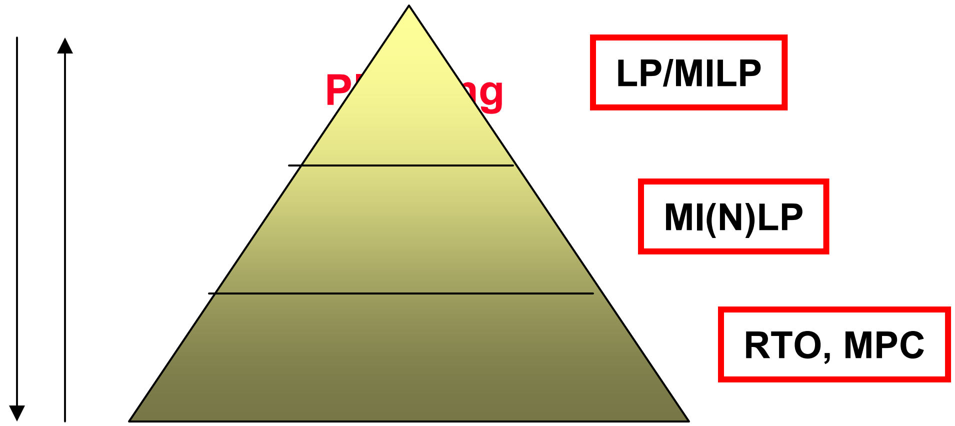
- ## Pharmaceutical process *(Shah, 2003)*
  - Primary production has five synthesis stages
  - Two secondary manufacturing
  - Global market



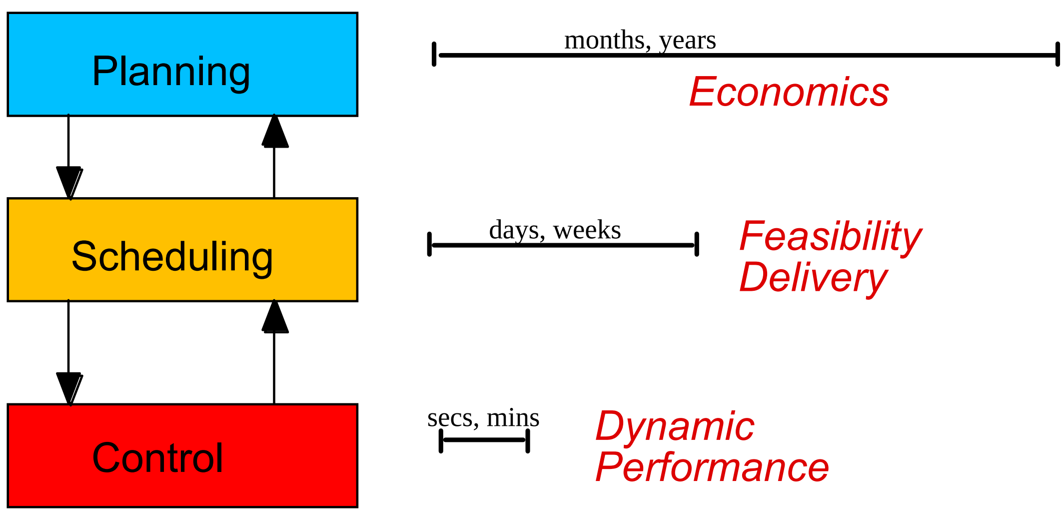
Key issues:

I. Integration of planning, scheduling and control

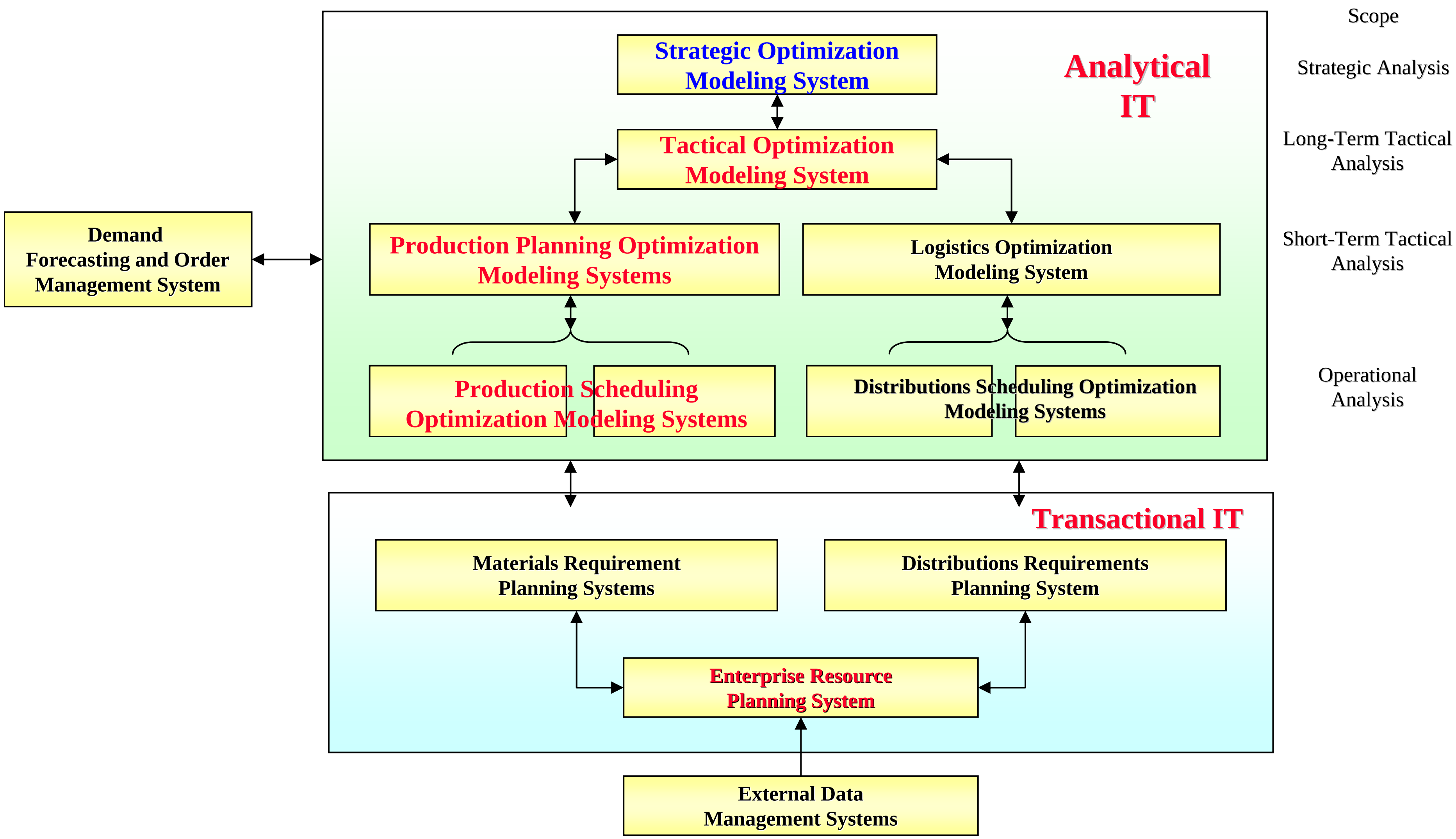
Mutiple models



Mutiple time scales



II. Integration of information, modeling and solution methods



Source: Tayur, et al. [1999]

Research Challenges

-The modeling challenge:
Planning, scheduling, control models **for the various components of the supply chain, including** ***nonlinear process*** **models?**

- The multi-scale optimization challenge:
Coordinated planning/scheduling models over ***geographically distributed*** **sites, and over the** ***long-term*** **(years),** ***medium-term*** **(months) and** ***short-term*** **(days, min) decisions?**

- The uncertainty challenge:
How to effectively anticipate effect of ***uncertainties*** **?**

- Algorithmic and computational challenges:
How to ***effectively solve large-scale*** **models including** ***nonconvex problems*** **in terms of efficient algorithms, decomposition methods and modern computer architectures?**

Examples of EWO problems

Multiperiod Supply Chain Design

Multiperiod mixed-integer linear programming model

Supply Chain Operation under Uncertainty

Two-stage programming LP model

Design of Responsive Process Supply Chains with Uncertain Demand

Bi-criterion mixed-integer nonlinear programming

Simultaneous Tactical Planning and Production Scheduling

Large-scale mixed integer linear programming

Optimal Planning of Multisite Distribution Network

Lagrangean decomposition for nonlinear programming model

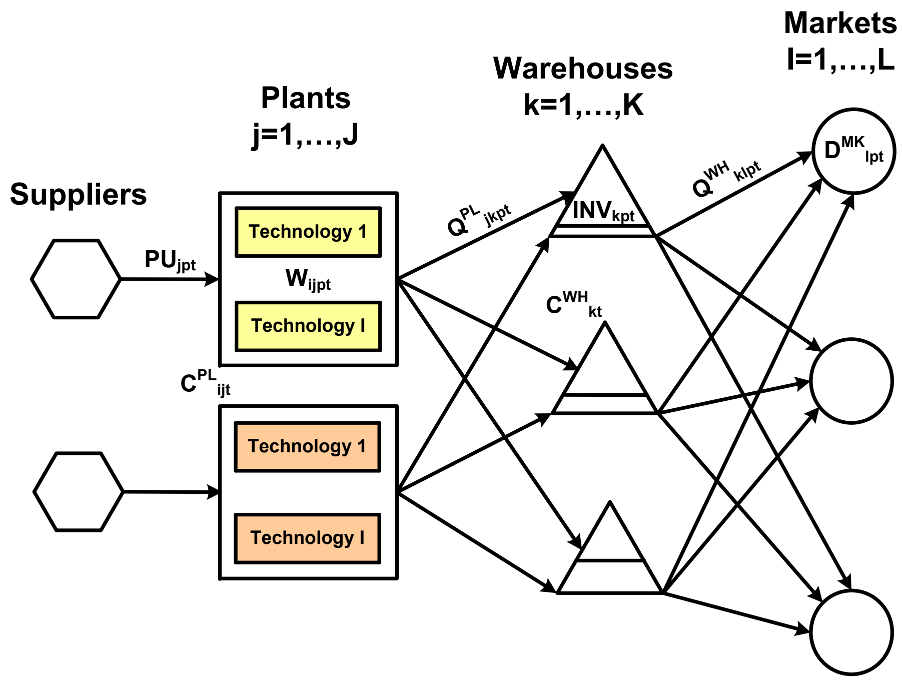
Supply Chain Design with Stochastic Inventory Management

Lagrangean decomposition for mixed-integer nonlinear programming model

Multiperiod Supply Chain Design and Planning

Guillen, Grossmann (2008)

- Three-echelon supply chain
- Different technologies available at plants
- Multi-period model



Model = Plant location problem (Current et al.,1990) plus
Long range planning of chemical processes (Sahinidis et al., 1989)

Notation

j	plants
k	warehouses
l	markets
p	products
t	time periods

Variables

C_{ijt}^{PL}	capacity of manufacturing technology i at plant j in time period t
CE_{ijt}^{PL}	capacity expansion of manufacturing technology i at plant j in time period t
C_{kt}^{WH}	capacity of warehouse k in time period t
CE_{kt}^{WH}	capacity expansion of warehouse k in time period t
CF_t	cash flow in period t
FCI	fixed capital investment
$FTDC_t$	fraction of the total depreciable capital that must be paid in period t
IL_{kt}	average inventory level at warehouse k in time period t
NE_t	net earnings in period t
NPV	net present value
PU_{jpt}	purchases of product p made by plant j in period t
Q_{jkpt}^{PL}	flow of product p sent from plant j to warehouse k in period t
Q_{klpt}^{WH}	flow of product p sent from warehouse k to market l in period t
SA_{lpt}	sales of product p at market l in time period t
W_{ijpt}	input/output flow of product p associated with technology i at plant j in t
X_{ijt}^{PL}	binary variable (1 if the capacity of manufacturing technology i at plant j is expanded in time period t, 0 otherwise)
X_{kt}^{WH}	binary variable (1 if the capacity of warehouse k is expanded in time period t, 0 otherwise)
Y_{jkt}^{PL}	binary variable (1 if a transportation link between plant j and warehouse k is established in time period t, 0 otherwise)
Y_{klt}^{WH}	binary variable (1 if a transportation link between warehouse k and market l is established in time period t, 0 otherwise)

Multiperiod MILP formulation (I)

1. Mass balances

$$PU_{jpt} + \sum_{i \in OUT(p)} W_{ijpt} = \sum_{k} Q^{PL}_{jkpt} + \sum_{i \in IN(p)} W_{ijpt} \qquad \forall j,p,t$$

$$W_{ijpt} = \mu_{ip} W_{ijp't} \qquad \forall i,j,p,t \qquad \forall p' \in MP(i)$$

Plants

$$INV_{kpt-1} + \sum_{j} Q^{PL}_{jkpt} = \sum_{l} Q^{WH}_{klpt} + INV_{kpt} \qquad \forall k,p,t$$

Warehouses

$$\underline{D^{MK}_{lpt}} \le \sum_{k} Q^{WH}_{klpt} \le \overline{D^{MK}_{lpt}} \qquad \forall l,p,t$$

Markets

2. Capacity Expansion Plants

$$W_{ijpt} \le C^{PL}_{ijt} \qquad \forall i,j,t \qquad \forall p \in MP(i)$$

$$C^{PL}_{ijt} = C^{PL}_{ijt-1} + CE^{PL}_{ijt} \qquad \forall i,j,t$$

$$\underline{CE^{PL}_{ijt}} X^{PL}_{ijt} \le CE^{PL}_{ijt} \le \overline{CE^{PL}_{ijt}} X^{PL}_{ijt} \qquad \forall i,j,t$$

Plants

Binary variable (1 if technology i is expanded in plant j in period t)

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Multiperiod MILP formulation (II)

3. Capacity Expansion Warehouses

$$\sum_{p} INV_{kpt} \leq C_{kt}^{WH} \qquad \forall k,t$$

$$C_{kt}^{WH} = C_{kt-1}^{WH} + CE_{kt}^{WH} \qquad \forall k,t$$

$$\underline{CE_{kt}^{WH}} X_{kt}^{WH} \leq CE_{kt}^{WH} \leq \overline{CE_{kt}^{WH}} X_{kt}^{WH} \qquad \forall k,t$$

Binary variable (1 if warehouse k is expanded in period t)

Warehouses

4. Transportation links

$$\underline{Q_{jkt}^{PL}} Y_{jkt}^{PL} \leq Q_{jkt}^{PL} \leq \overline{Q_{jkt}^{PL}} Y_{jkt}^{PL} \qquad \forall j,k,t$$

Binary variable (1 if there is a transport link between plant j and warehouse k in period t)

$$\underline{Q_{klt}^{WH}} Y_{klt}^{WH} \leq Q_{klt}^{WH} \leq \overline{Q_{klt}^{WH}} Y_{klt}^{WH} \qquad \forall k,l,t$$

Binary variable (1 if there is a transport link between warehouse k and market l in period t)

Transport links

Multiperiod MILP formulation (III)

5. Objective function

$$NPV = \sum_t \frac{CF_t}{(1+ir)^{t-1}}$$

Summation of discounted cash flows

$$CF_t = NE_t - FTDC_t \qquad t = 1, ..., NT-1$$

$$CF_t = NE_t - FTDC_t + SVFCI \qquad t = NT$$

Net Earnings

$$NE_t = (1-\varphi)\left[\sum_l \sum_p \gamma_{lpt}^{FP} SA_{lpt} - \sum_j \sum_p \gamma_{jpt}^{RM} PU_{jpt} - \sum_i \sum_j \sum_{p \in MP(i)} \upsilon_{ijpt} W_{ijpt} - \sum_k \pi_{kt} IL_{kt} - \sum_j \sum_k \sum_p \psi_{jkpt}^{PL} Q_{jkpt}^{PL} - \sum_k \sum_l \sum_p \psi_{klpt}^{WH} Q_{klpt}^{WH}\right] + \varphi DEP_t \qquad \forall t$$

$$DEP_t = \frac{(1-SV)FCI}{NT} \qquad \forall t$$

Fixed cost

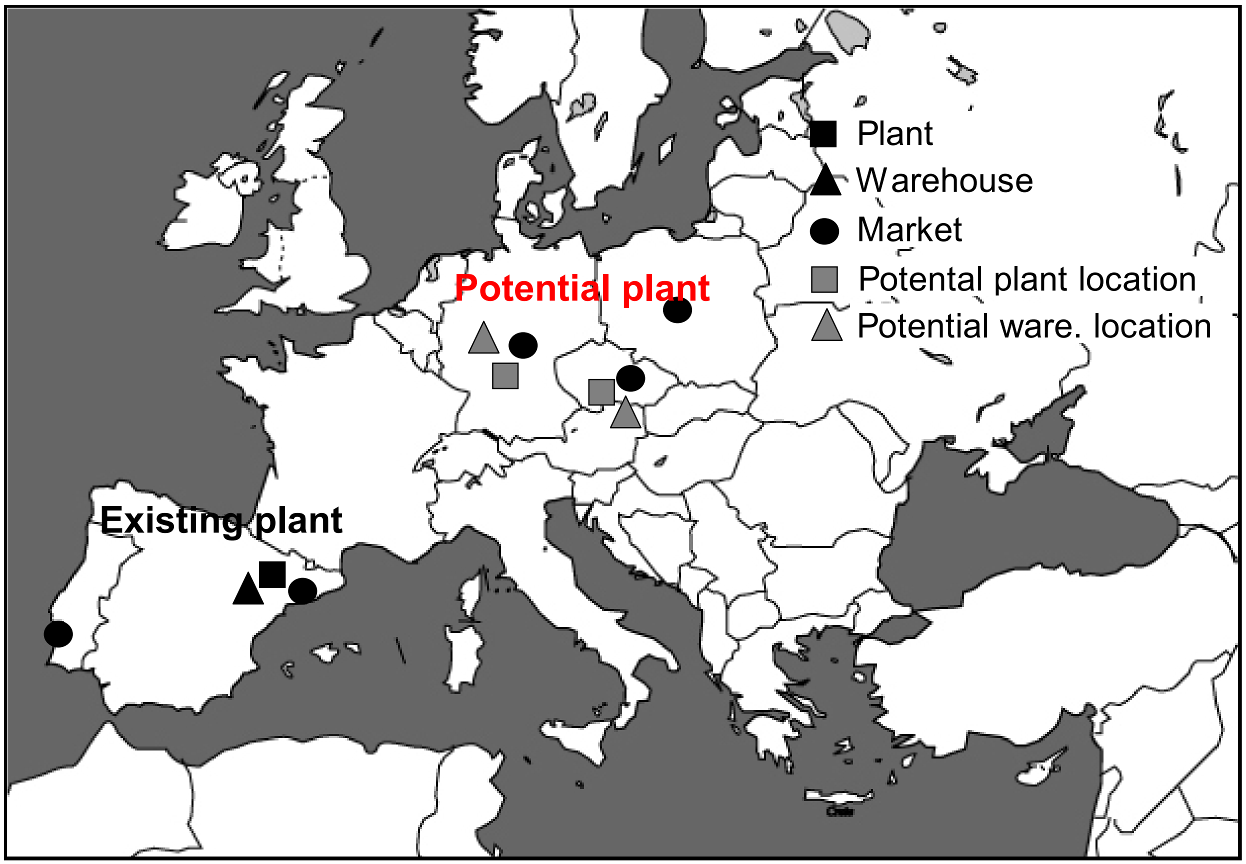
$$FCI = \sum_i \sum_j \sum_t \left(\alpha_{ijt}^{PL} CE_{ijt}^{PL} + \beta_{ijt}^{PL} X_{ijt}^{P}\right) + \sum_k \sum_t \left(\alpha_{kt}^{WH} CE_{kt}^{WH} + \beta_{kt}^{WH} X_{kt}^{WH}\right)$$

$$\sum_j \sum_k \sum_t \left(\beta_{jkt}^{TPL} Y_{jkt}^{PL}\right) + \sum_k \sum_l \sum_t \left(\beta_{klt}^{TWH} Y_{klt}^{WH}\right)$$

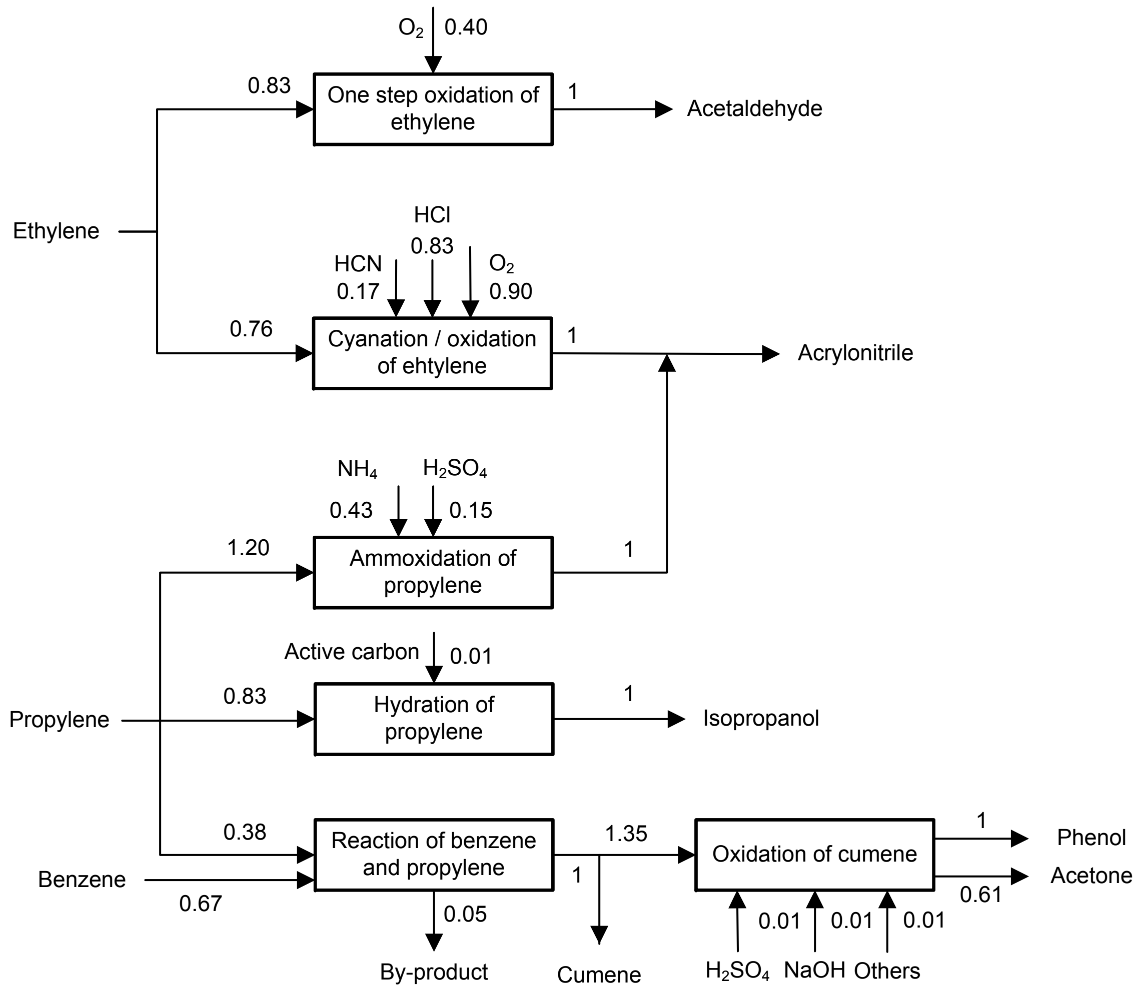
Case study

Problem :

- Redesign a petrochemical SC to fulfill future forecasted demand

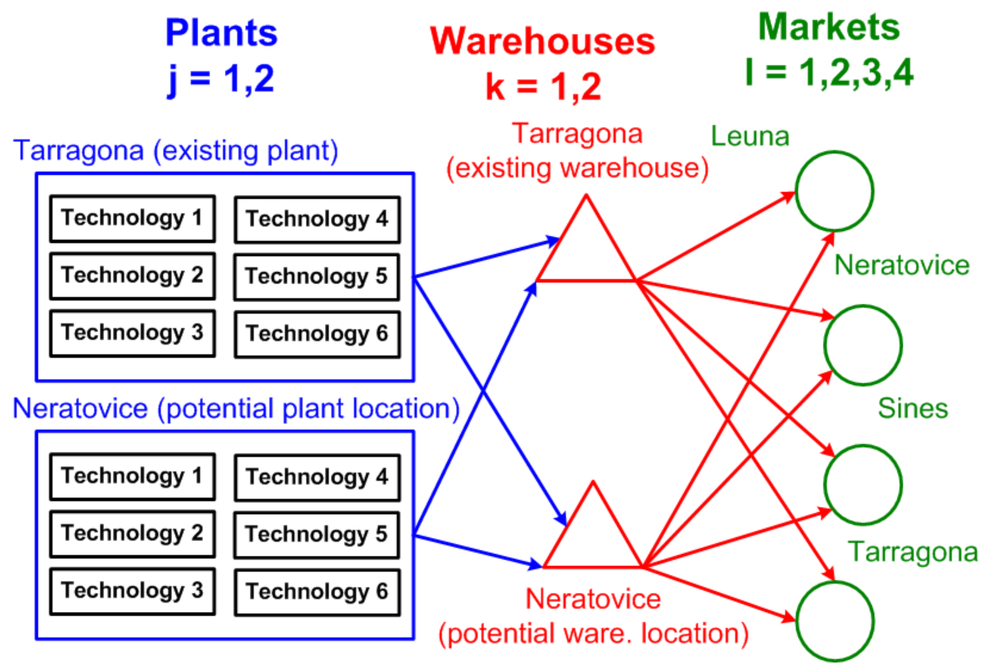


Technologies in each Plant Site



Potential Supply Chain

Horizon: 3 yrs



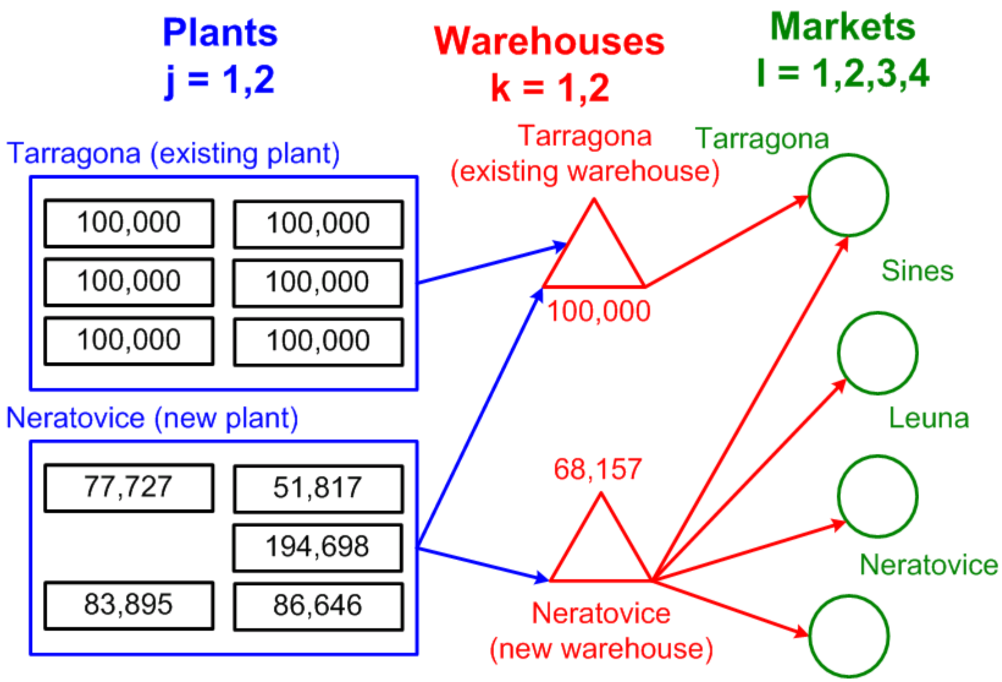
Multiperiod MILP Models:

- **Number of 0-1 variables: 450**
- **Number of continuous variables: 4801**
- **Number of equations: 4682**
- **CPU* time: 0.33 seconds**

*Solved with GAMS 21.4 / CPLEX 9.0 (Pentium 1.66GHz)

Optimal Solution

NPV = $132 million

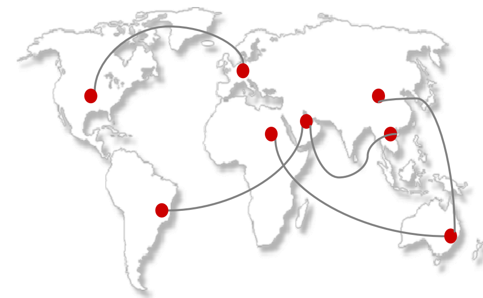


Supply Chain Operation under Uncertainty

You, Grossmann, Wassick (2008)

Chemical Supply chain: an integrated network of business units for the supply, production, distribution and consumption of the products.

Case Study



- Given
  - Minimum and initial inventory
  - Inventory holding cost and throughput cost
  - Transport times of all the transport links & modes
  - Uncertain customer demands and transport cost
- Determine
  - Transport amount, inventory and production levels

• Objective: Minimize Cost & Risks

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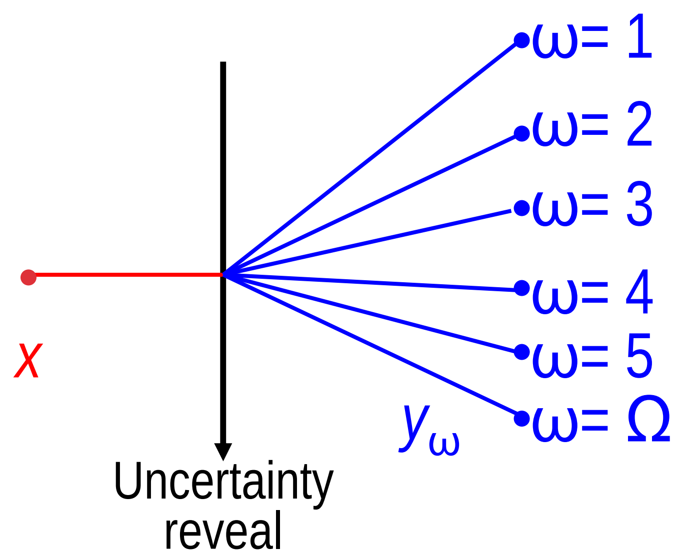
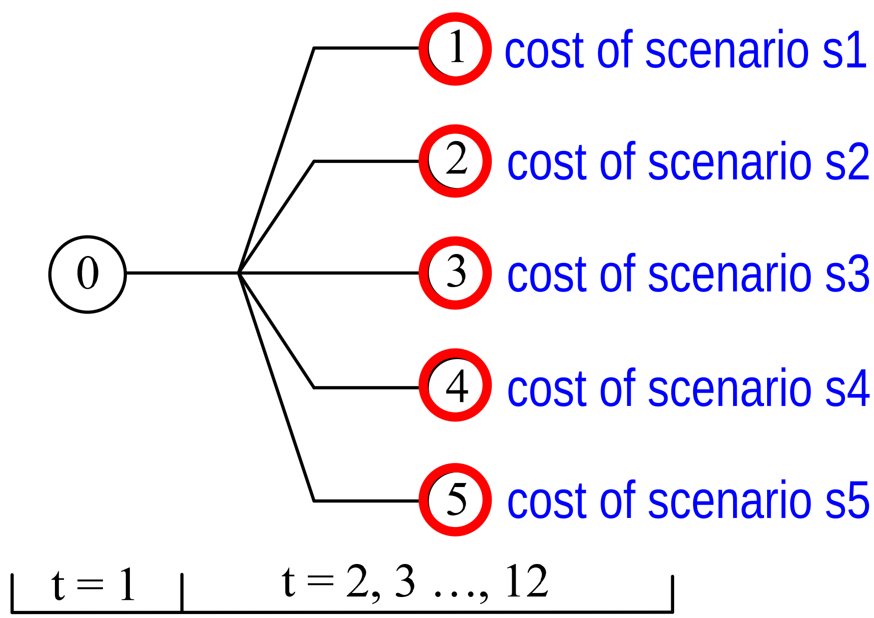
Stochastic Programming

- Scenario Planning
  - A scenario is a future possible outcome of the uncertainty
  - Find a solution perform well for all the scenarios
- Two-stage Decisions
  - Here-and-now: Decisions (x) are taken before uncertainty ω reveals
  - Wait-and-see: Decisions (y_ω) are taken after uncertainty ω reveals as "*corrective* action" - ***recourse***

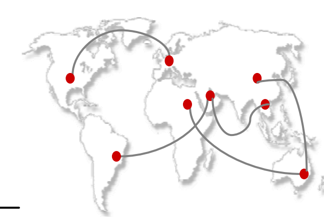


Stochastic Programming for Case Study

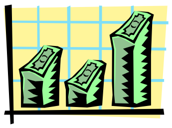
- First stage decisions
  - Here-and-now: decisions for the first month (production, inventory, shipping)
- Second stage decisions
  - Wait-and-see: decisions for the remaining 11 months



Minimize E [cost]



Objective Function



$$E[Cost] = Cost1 + \sum_s P_s \cdot Cost2_s$$

First stage cost

Probability of each scenario

Second stage cost

$Cost1 =$

$$\sum_k \sum_j \sum_t h_{k,j} I_{k,j,t}$$

$$+ \sum_k \sum_{k'} \sum_j \sum_t \gamma_{k,k',j} F_{k,k',j,t}$$

$$+ \sum_k \sum_l \sum_j \sum_t \gamma_{k,l,j} S_{k,l,j,t}$$

$$+ \sum_k \sum_{k'} \sum_j \sum_t \delta_{k,j} F_{k,k',j,t}$$

$$+ \sum_k \sum_l \sum_j \sum_t \delta_{k,j} S_{k,l,j,t}$$

Inventory Costs

Freight Costs

Throughput Costs

Demand Unsatisfied

$Cost2_s =$

$$\sum_k \sum_j \sum_t h_{k,j} I_{k,j,t,s}$$

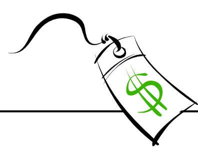
$$+ \sum_k \sum_{k'} \sum_j \sum_t \gamma_{k,k',j,s} F_{k,k',j,t,s}$$

$$+ \sum_k \sum_l \sum_j \sum_t \gamma_{k,l,j,s} S_{k,l,j,t,s}$$

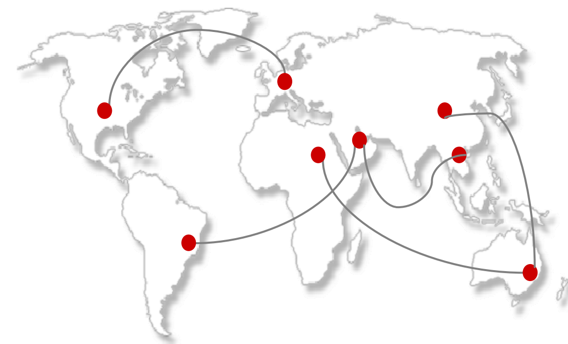
$$+ \sum_k \sum_{k'} \sum_j \sum_t \delta_{k,j} F_{k,k',j,t,s}$$

$$+ \sum_k \sum_l \sum_j \sum_t \delta_{k,j} S_{k,l,j,t,s}$$

$$+ \sum_l \sum_j \sum_t \eta_{l,j} SF_{l,j,t,s}$$

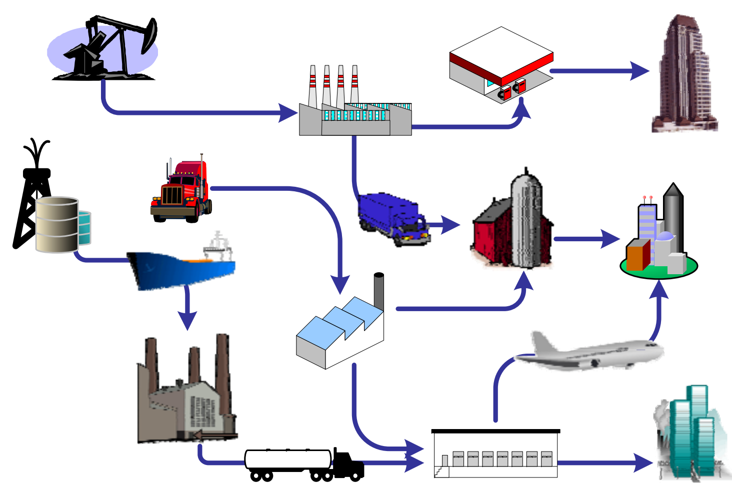


Multiperiod Planning Model (Case Study)

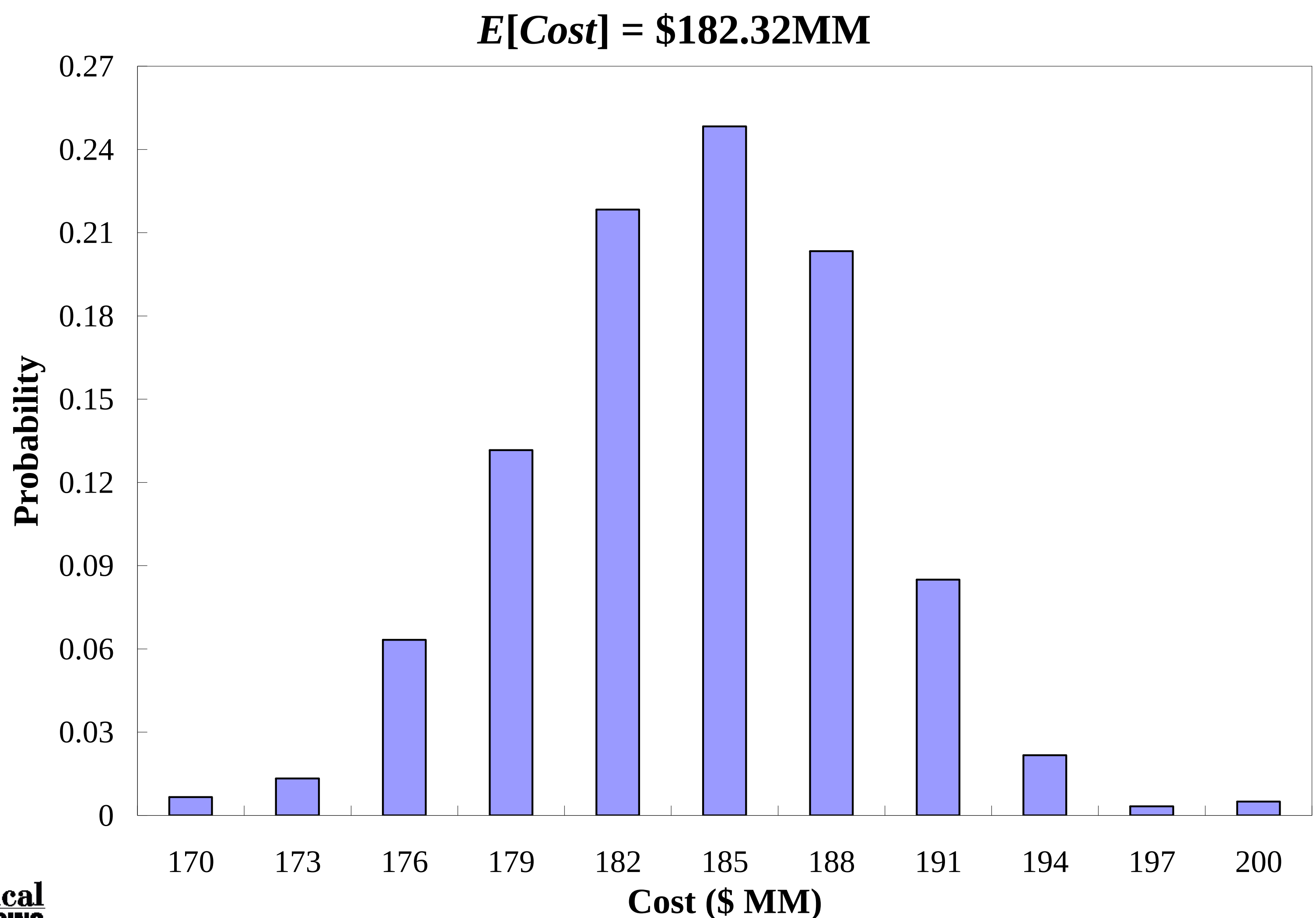


- Objective Function:
  - Min: Total Expected Cost

- Constraints:
  - Mass balance for plants
  - Mass balance for DCs
  - Mass balance for customers
  - Minimum inventory level constraint
  - Capacity constraints for plants



Result of Two-stage SP Model



Problem Sizes

Small Problem	Deterministic Model	Two-stage Stochastic Programming Model		
		10 scenarios	100 scenarios	1,000 scenarios
# of Constraints	1,369	13,080	130,170	1,301,070
# of Variables	3,937	37,248	370,338	3,701,240
# of Non-zeros	8,910	85,451	850,271	8,498,429

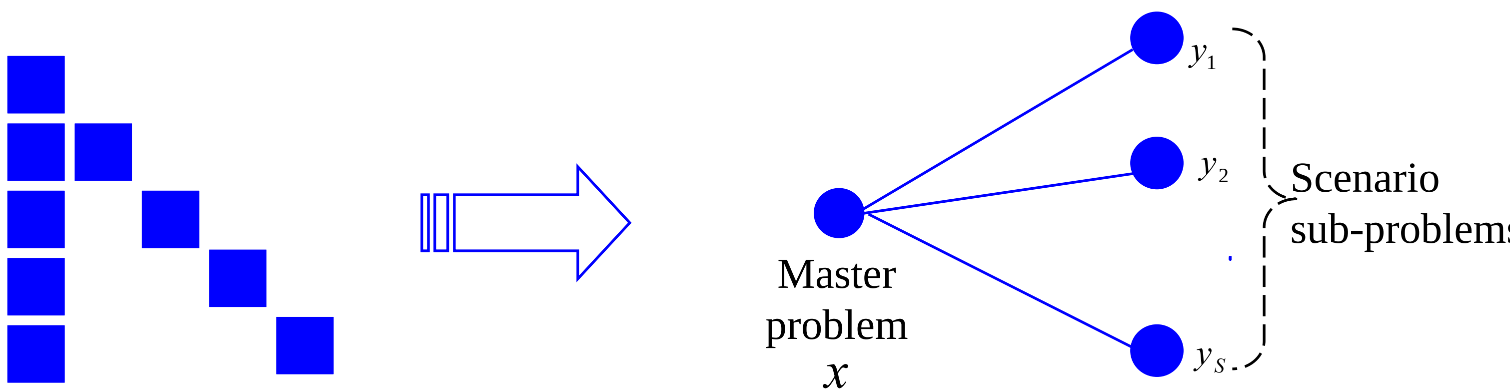
Full Problem	Deterministic Model	Two-stage Stochastic Programming Model		
		10 scenarios	100 scenarios	1,000 scenarios
# of Constraints	6,373	61,284	610,374	6,101,280
# of Variables	19,225	182,496	1,815,816	18,149,077
# of Non-zeros	41,899	402,267	4,004,697	40,028,872

Note: Problems with red statistical data are not able to be solved by DWS

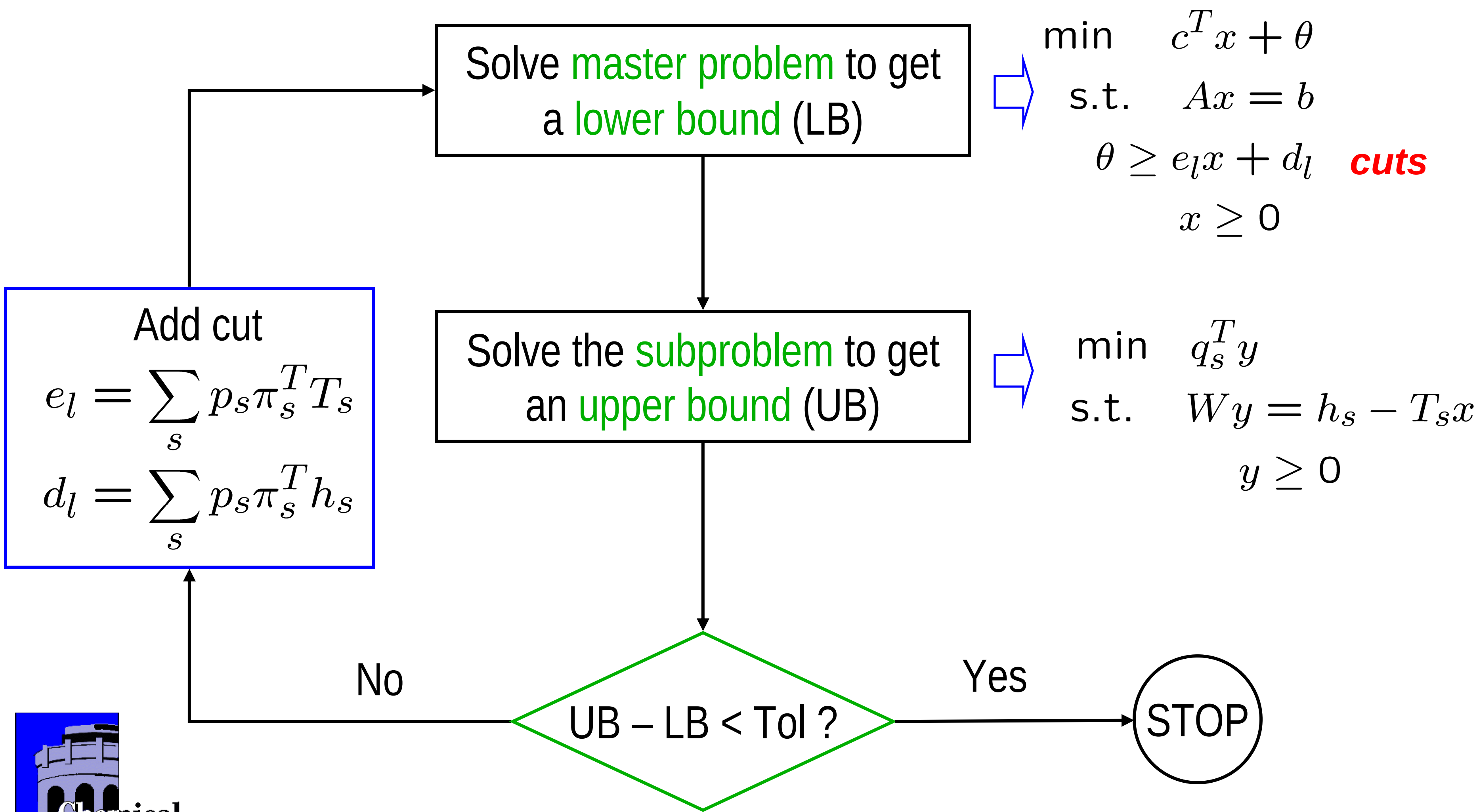
Two-stage SP Model

$$
\begin{array}{llcl}
\text{Min} & c^T x & + & p_1 q_1^T y_1 + p_2 q_2^T y_2 \; \cdot \;\; \cdot \;\; \cdot + p_s q_s^T y_s \\
\text{s.t.} & Ax & & \hspace{10em} = b \;\rightarrow\; \text{Master problem} \\
& T_1 x & + & W_1 y_1 \hspace{10em} = h_1 \\
& T_2 x & + & \hspace{5em} W_2 y_2 \hspace{5em} = h_2 \\
& \vdots & + & \hspace{8em} \cdot \hspace{8em} = \;\vdots \\
& \vdots & + & \hspace{10em} \cdot \hspace{6em} = \;\vdots \\
& \vdots & + & \hspace{12em} \cdot \hspace{4em} = \;\vdots \\
& T_s x & + & \hspace{14em} W_s y_s = h_s
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{Scenario sub-problems}
$$

$$x \geq 0, \; y_1 \geq 0, \; y_2 \geq 0, \; \cdot \;\; \cdot \;\; \cdot \; y_s \geq 0$$



Standard L-shaped Method



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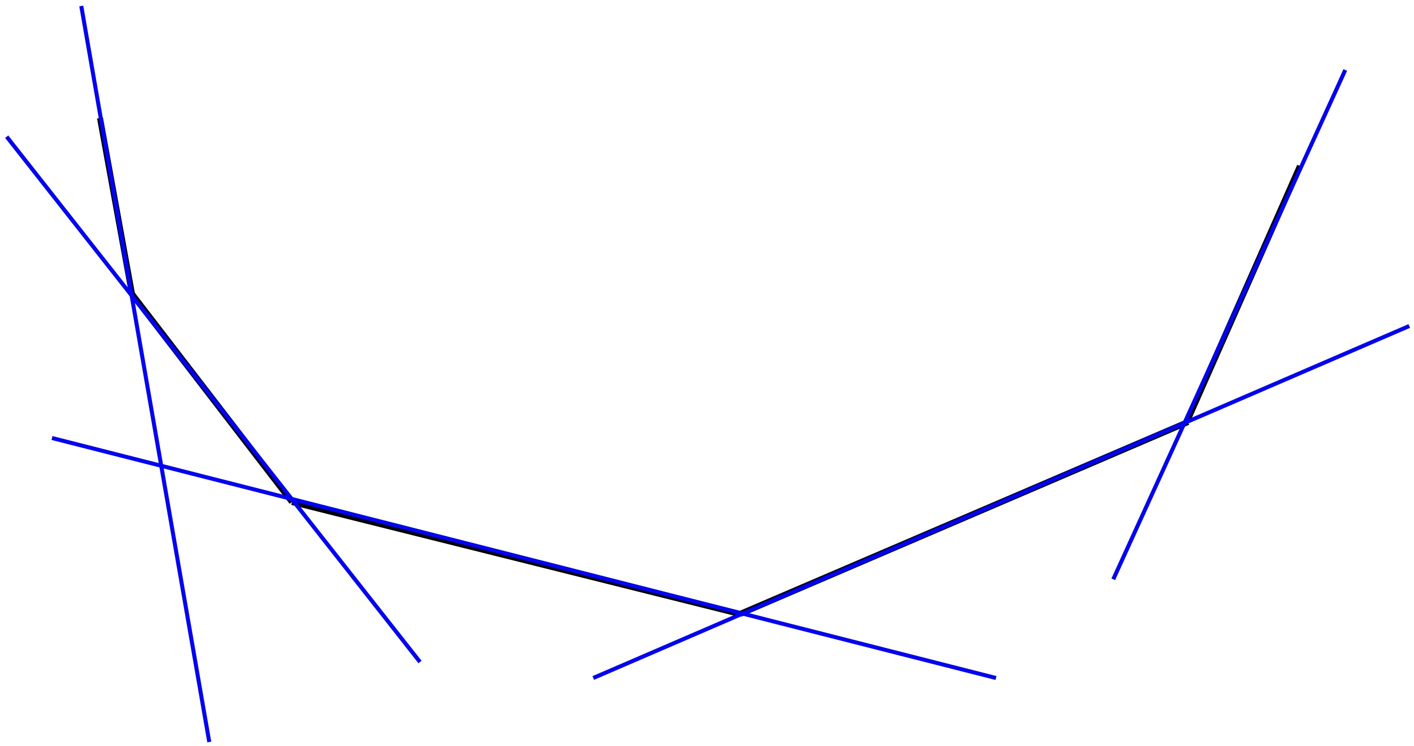
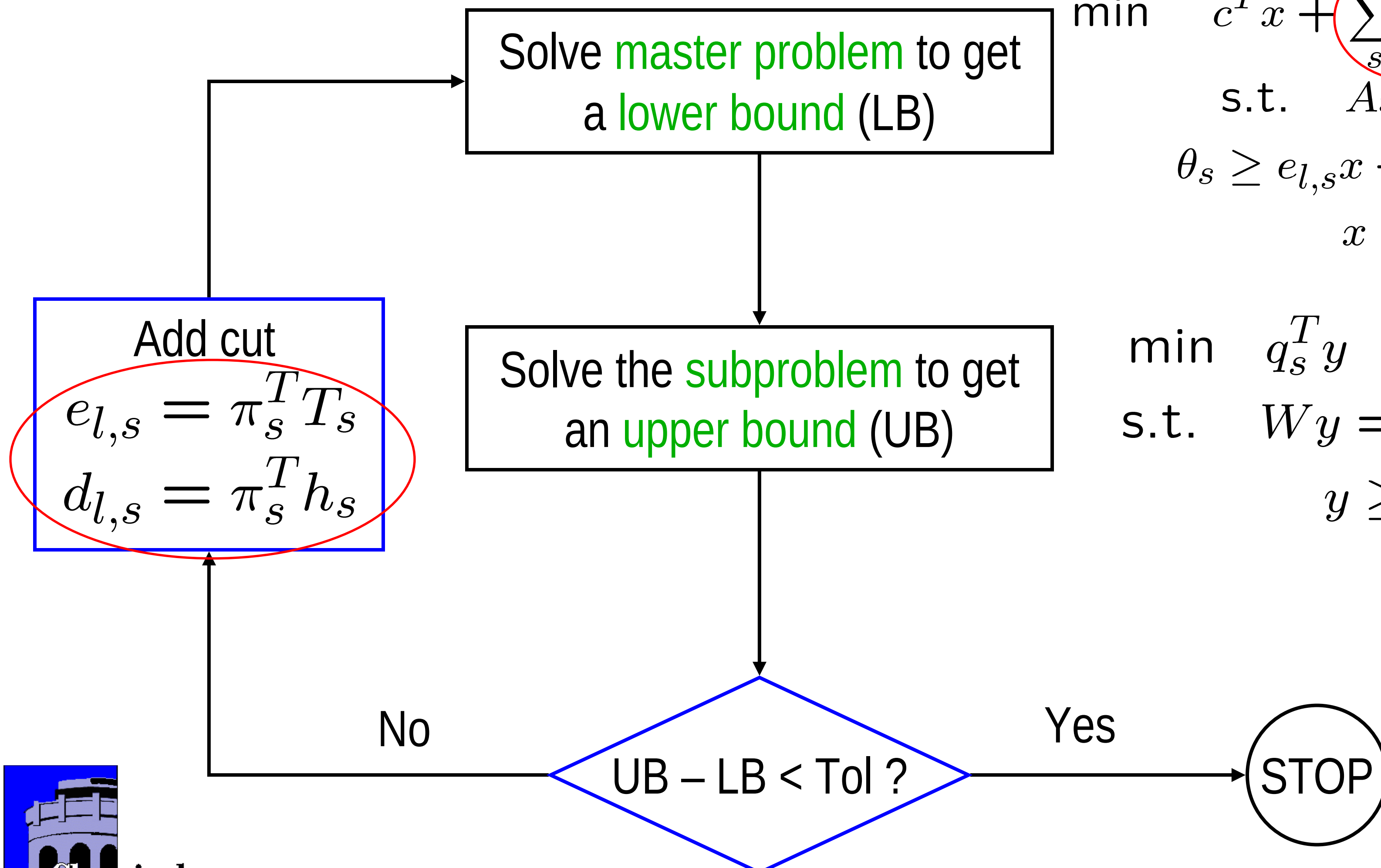
Expected Recourse Function



- The expected recourse function Q(x) is convex and piecewise linear
- Each optimality cut supports Q(x) from below

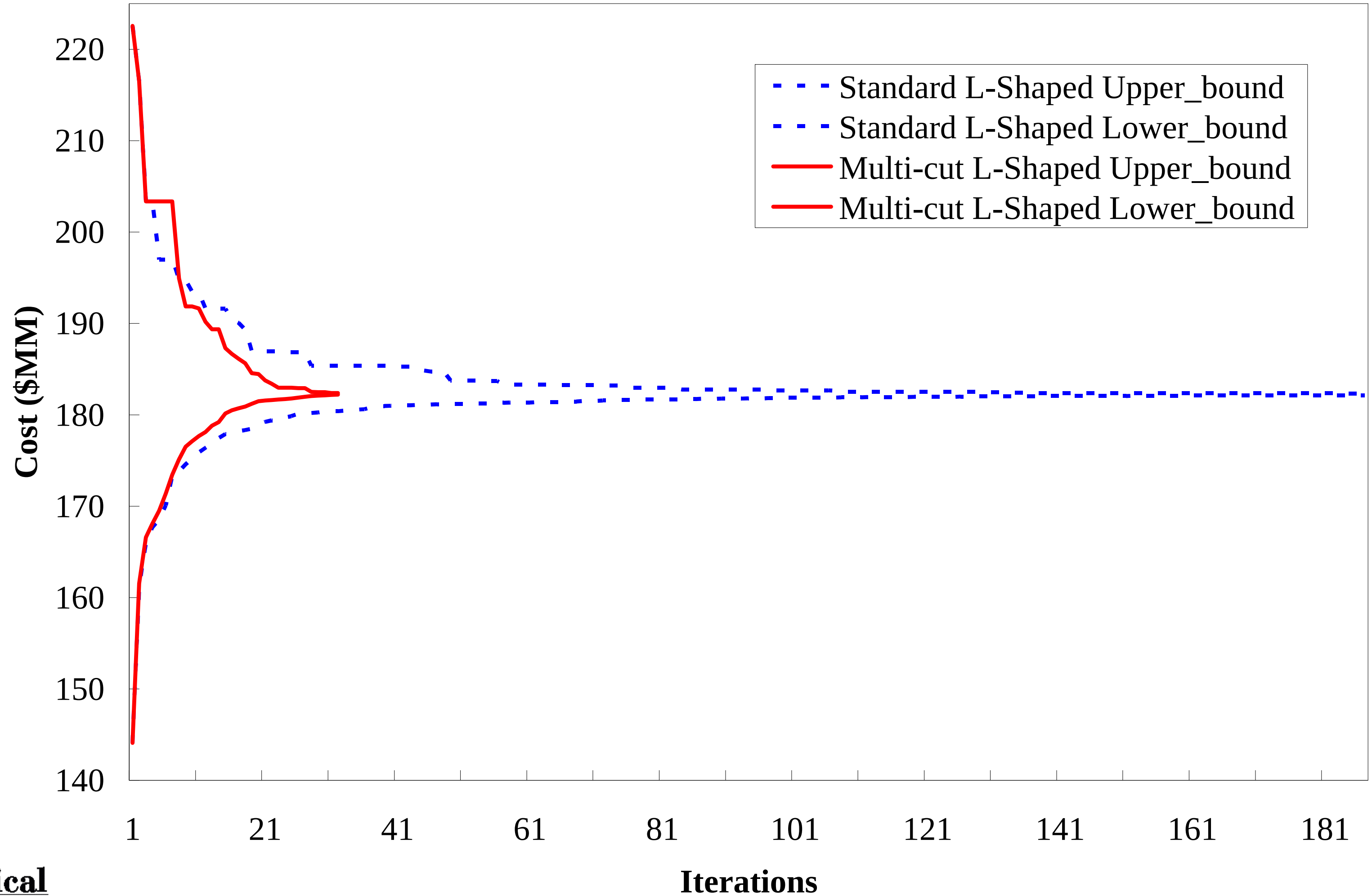
Multi-cut L-shaped Method



$$
\begin{aligned}
\min \quad & c^T x + \sum_s p_s \theta_s \\
\text{s.t.} \quad & Ax = b \\
& \theta_s \geq e_{l,s} x + d_{l,s} \quad \textbf{\textit{cuts}} \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & q_s^T y \\
\text{s.t.} \quad & Wy = h_s - T_s x \\
& y \geq 0
\end{aligned}
$$

Example


220
210
200
190
180
170
160
150
140
Cost ($MM)
1
21
41
61
81
101
121
141
161
181
Iterations
Standard L-Shaped Upper_bound
Standard L-Shaped Lower_bound
Multi-cut L-Shaped Upper_bound
Multi-cut L-Shaped Lower_bound

Optimal Design of Responsive Process Supply Chains

You, Grossmann (2008)

Objective: design supply chains under responsive and economic criteria with consideration of inventory management and demand uncertainty

Responsive Supply Chains

Problem Statement

Suppliers

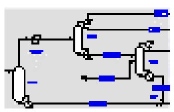
Plants

Where?
What?
When?

DCs

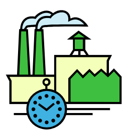
Customers

Production Network

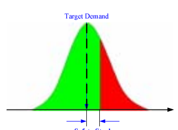


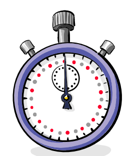
Costs and prices

Production and transportation time



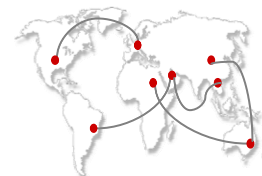
Demand information





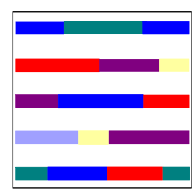
Max: Net present value

Max: Responsiveness



Network Structure

Operational Plan



Production Schedule

Production Network of Polystyrene Resins

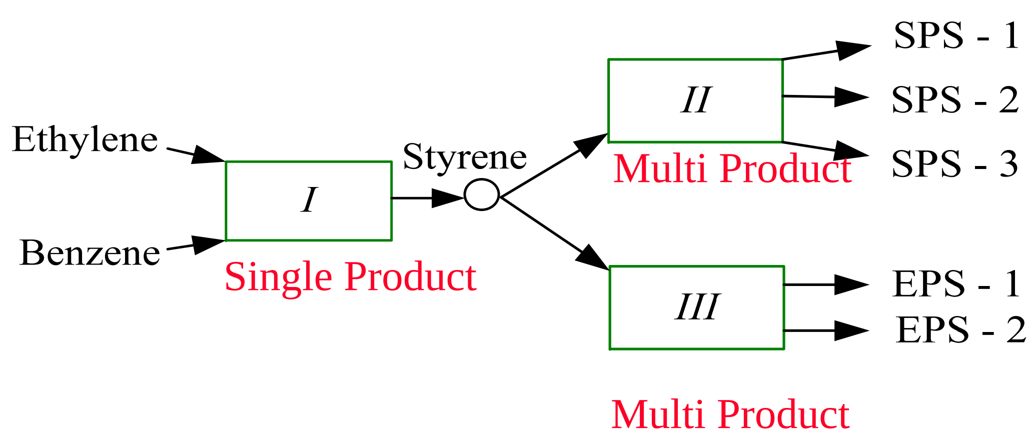
Three types of plants:

Plant *I*: *Ethylene + Benzene* ⟶ *Styrene* (1 products)

Plant *II*: *Styrene* ⟶ *Solid Polystyrene (SPS)* (3 products)

Plant *III*: *Styrene* ⟶ *Expandable Polystyrene (EPS)* (2 products)

Basic Production Network

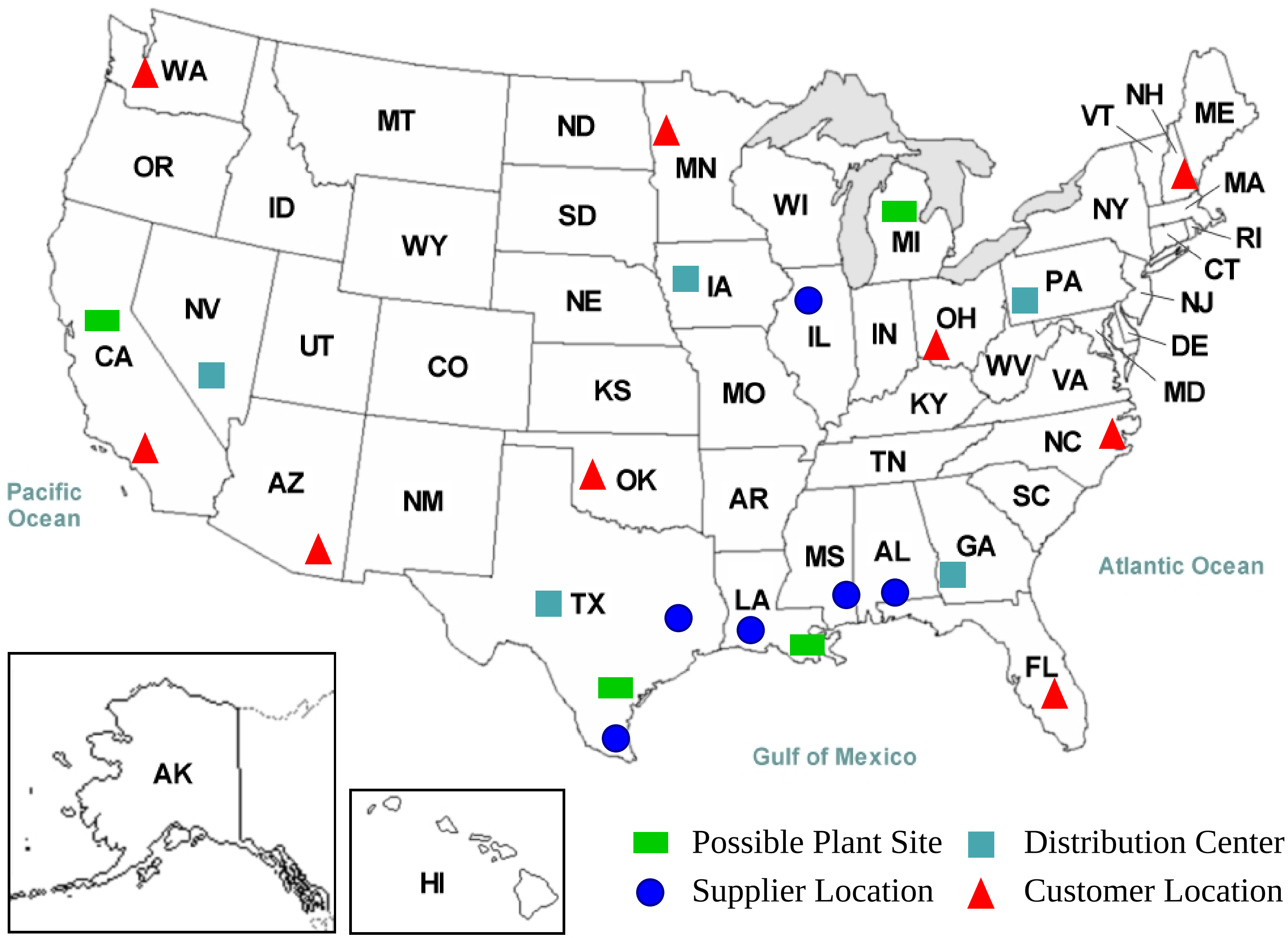


Source: Data Courtesy Nova Chemical Inc. http://www.novachem.com/

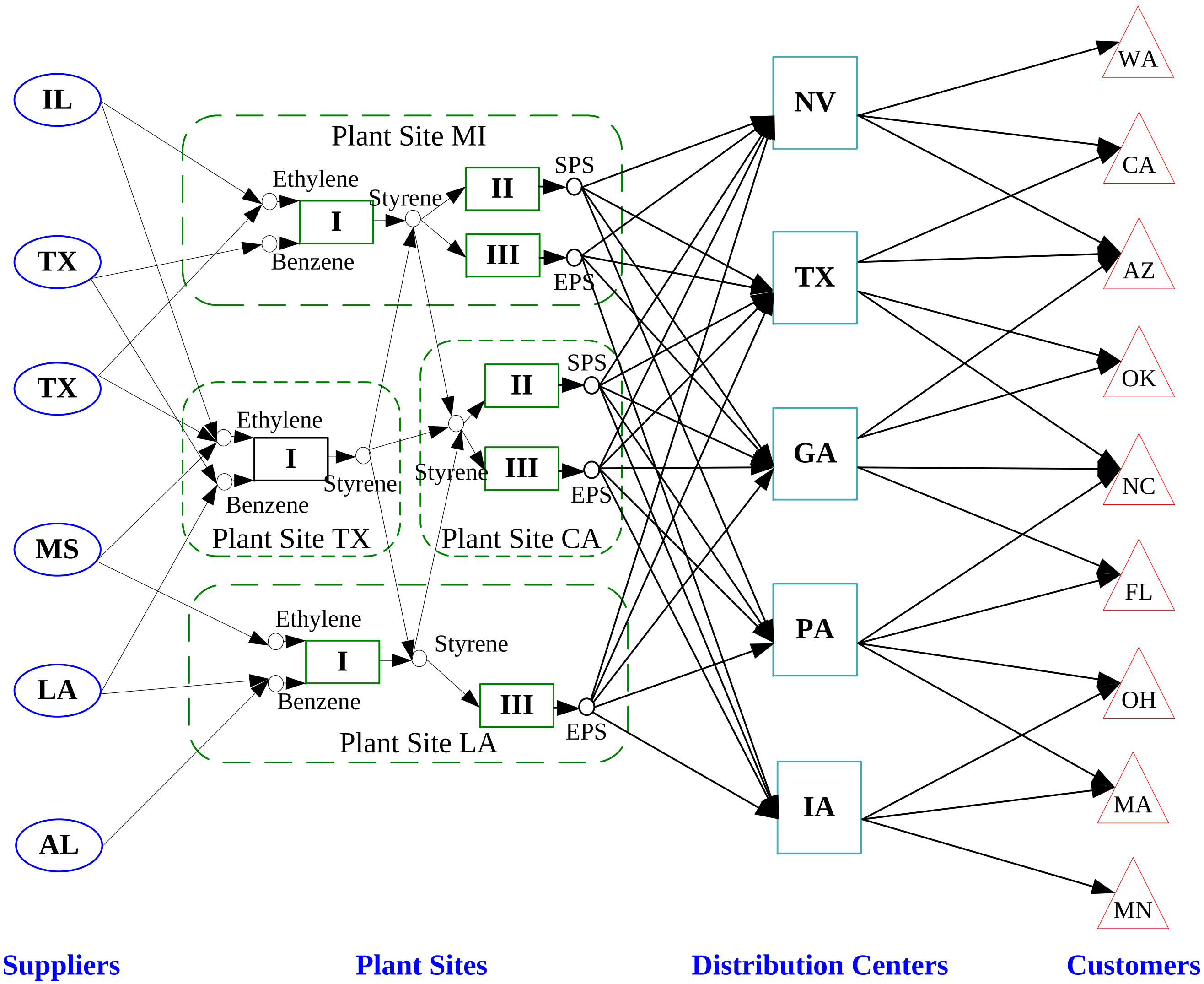
Responsive Supply Chains

CAPD CENTER

Location Map

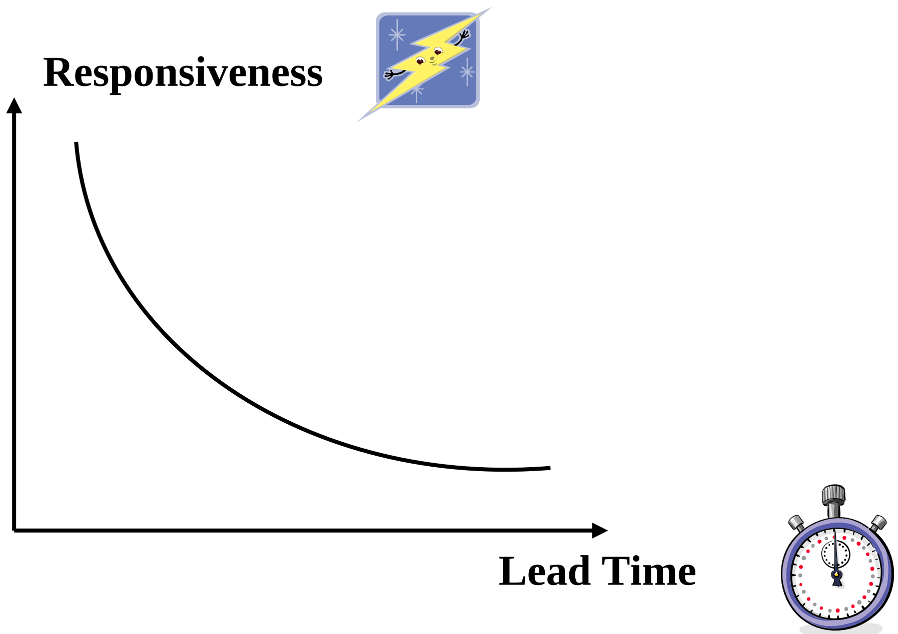


Potential Network Superstructure



Responsiveness - Lead Time

- Lead Time: The time of a supply chain network to respond to customer demands and preferences in the ***worst case***

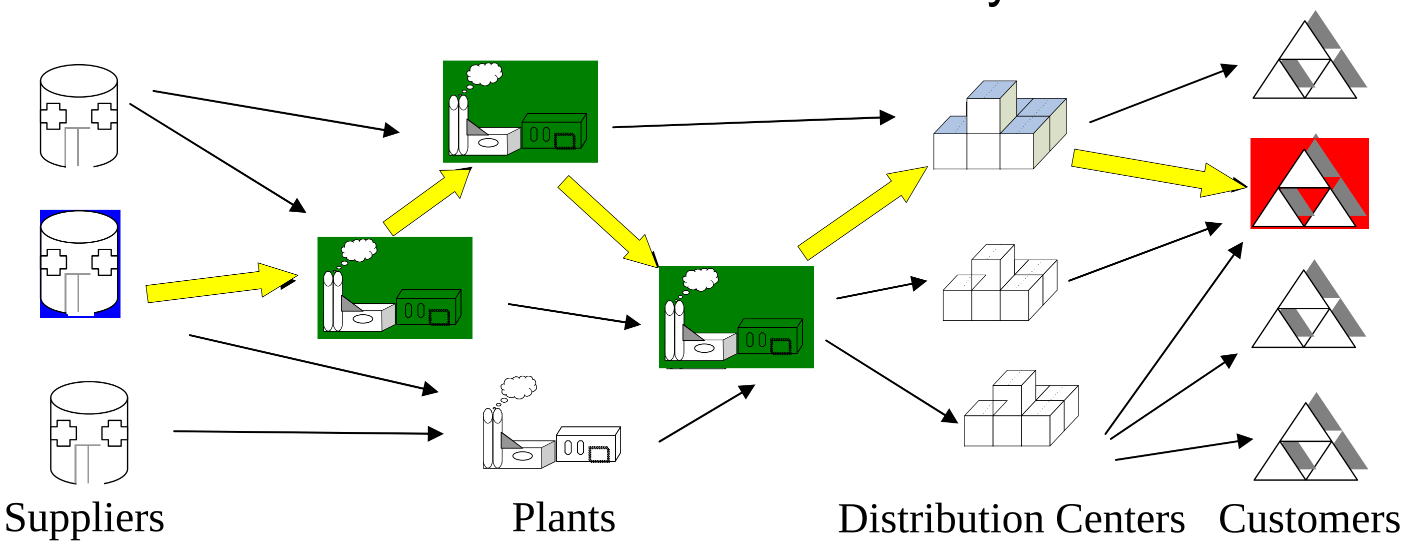


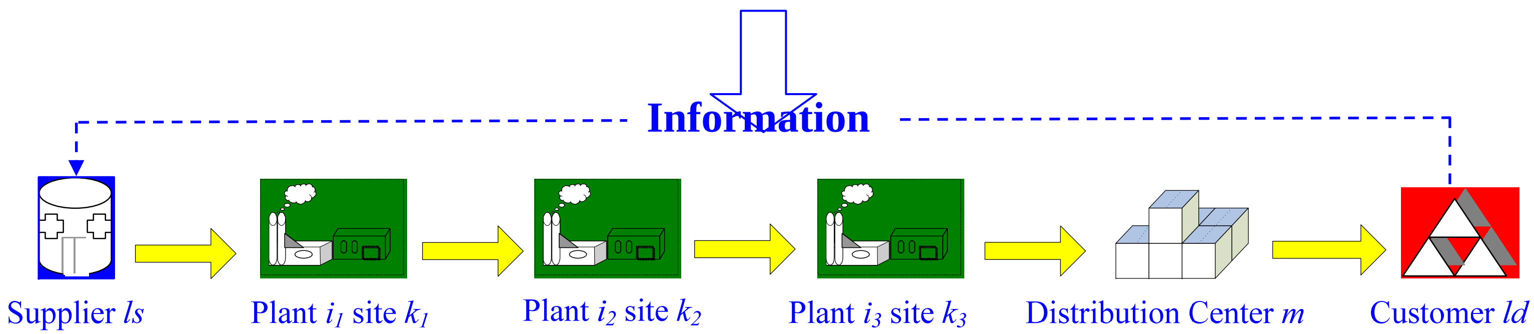
Lead Time is a measure of responsiveness in SCs

Responsive Supply Chains

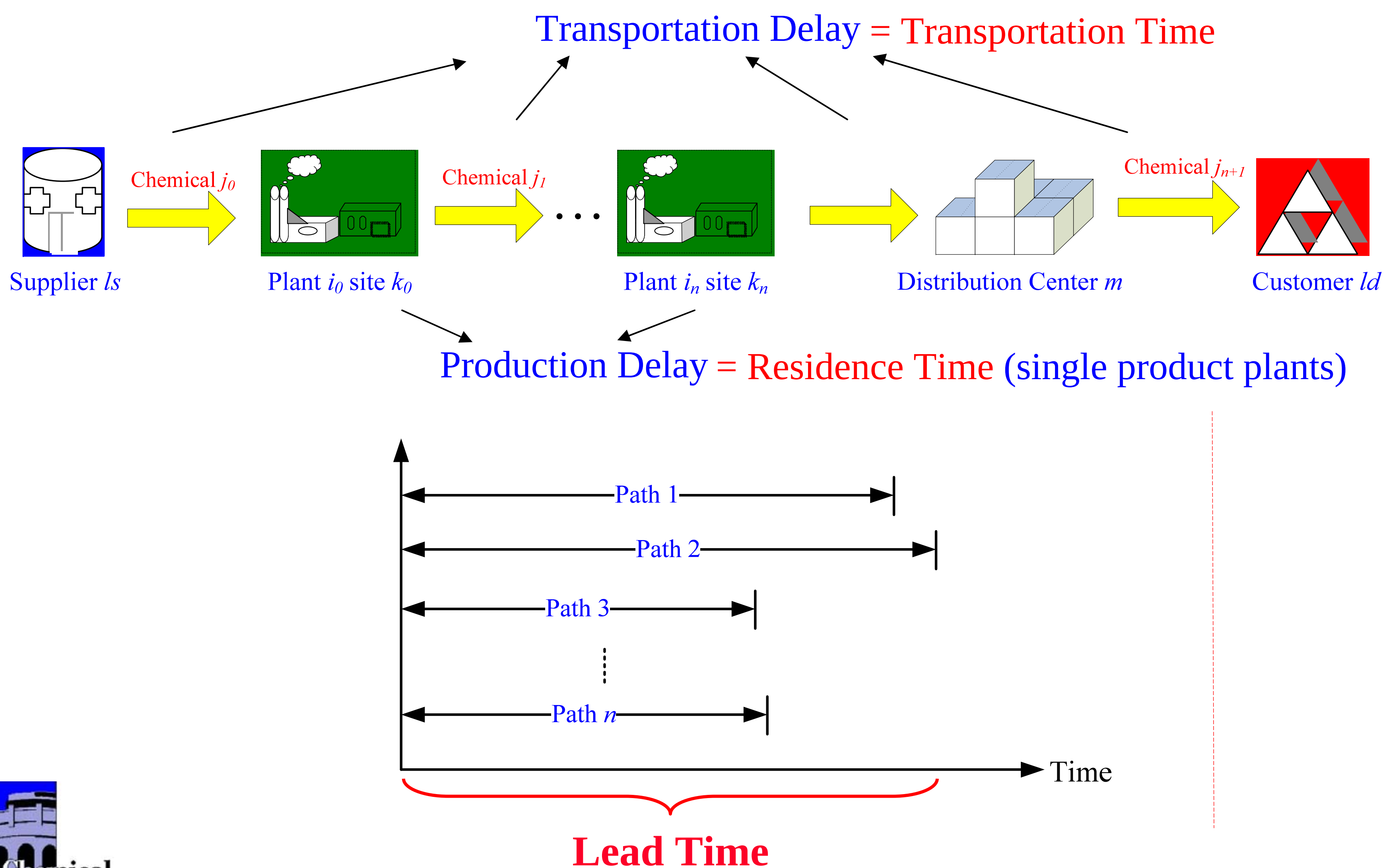
Lead Time for A Linear Supply Chain

- A supply chain network = ∑Linear supply chains
    - Assume information transfer instantaneously



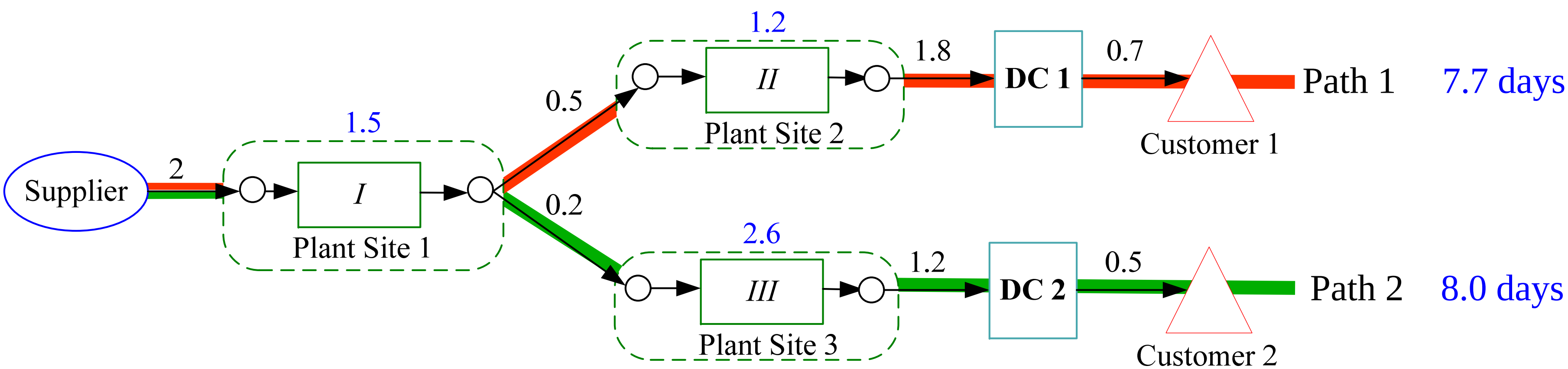


Lead Time for Deterministic Demand



Lead Time of SCN

- Lead time of a supply chain network (deterministic demand)
  - The longest lead time for all the paths in the network (***worst case***)
  - Example: A simple SC with all process are dedicated



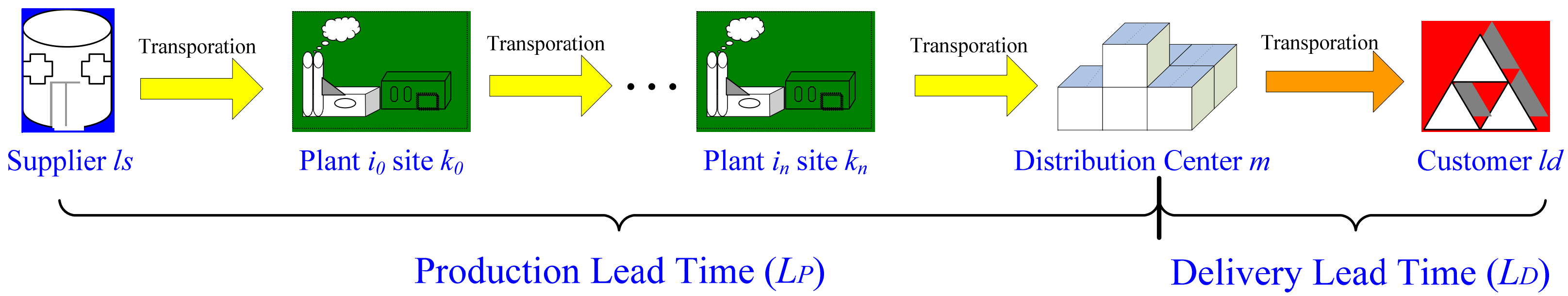
For Path 1: 2 + 1.5 + 0.5 + 1.2 + 1.8+ 0.7 = 7.7 days

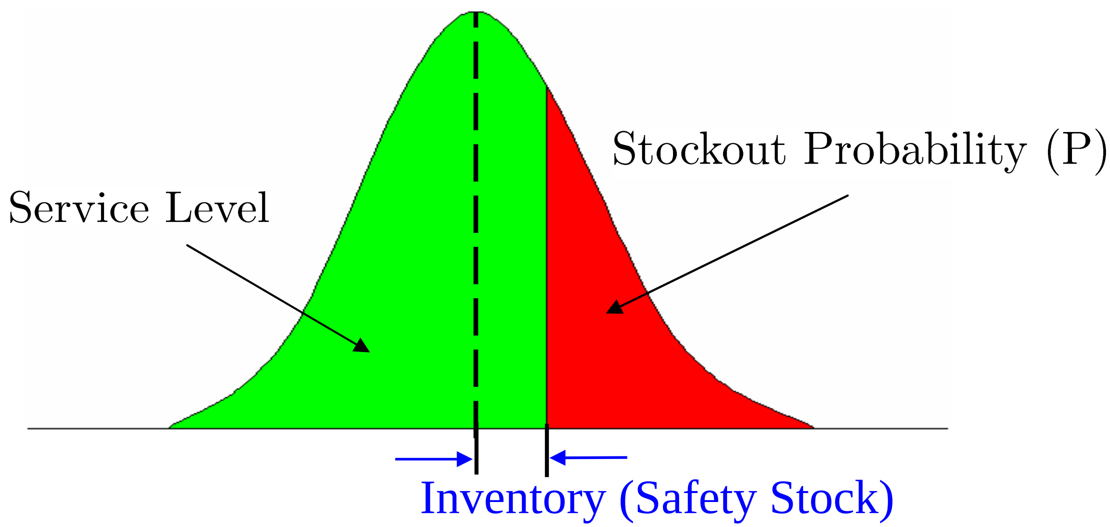
For Path 2: 2 + 1.5 + 0.2 + 2.6 + 1.2 + 0.5 = 8.0 days

Lead Time = max {7.7, 8.0} = 8.0 days

Responsive Supply Chains

Lead Time under Demand Uncertainty





$$\text{Expected Lead Time} = L_D + \text{P(Stockout)} \cdot L_P$$

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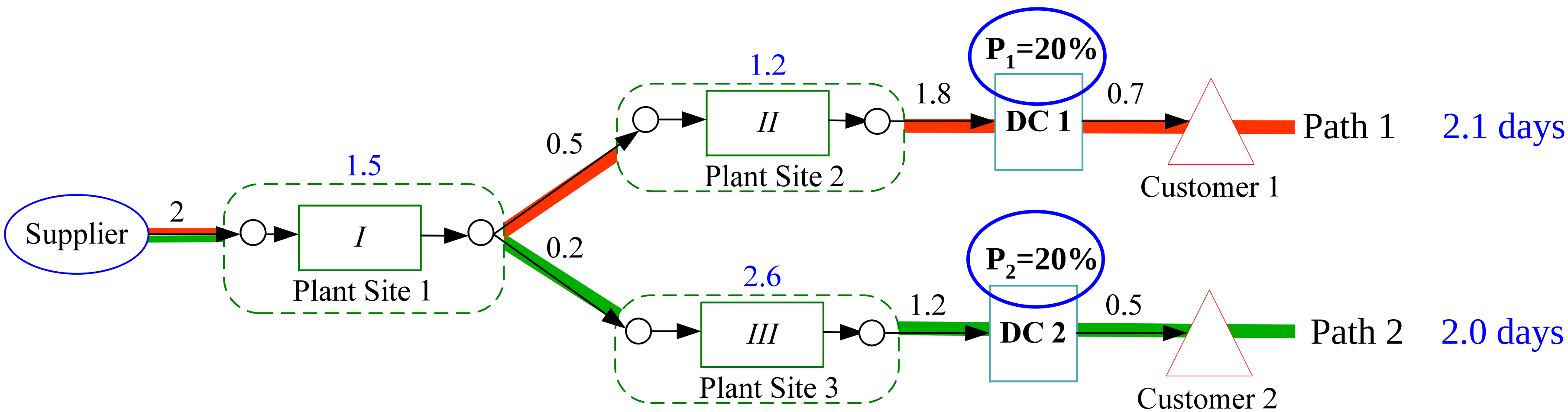
Expected Lead Time of SCN

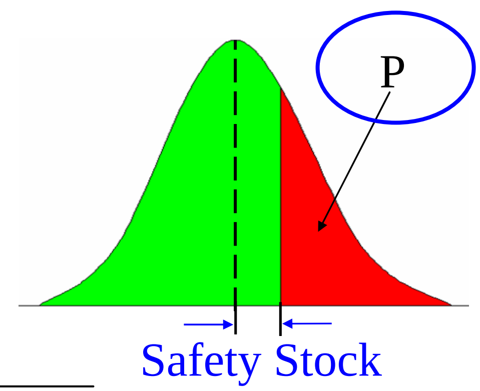
- Expected Lead time of a supply chain network (uncertain demand)
  - The longest expected lead time for all the paths in the network (***worst case***)
  - Example: A simple SC with all process are dedicated



For Path 1: (2 + 1.5 + 0.5 + 1.2 + 1.8)×**20%** + 0.7 = 2.1 days

For Path 2: (2 + 1.5 + 0.2 + 2.6 + 1.2)×**20%** + 0.5 = 2.0 days

Expected Lead Time = max {2.1, 2.0} = 2.1 days



Responsive Supply Chains

Stock-out Probability (P)

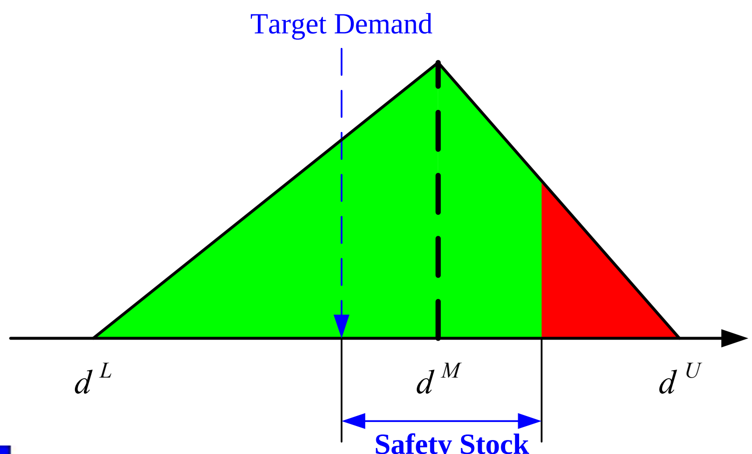
- Chance constraint for stockout probability
  - Integrate lead time, inventory management, demand uncertainty

$$Prob_{j,ld,t} = \Pr(QS_{j,ld,t} \leq d_{j,ld,t})$$

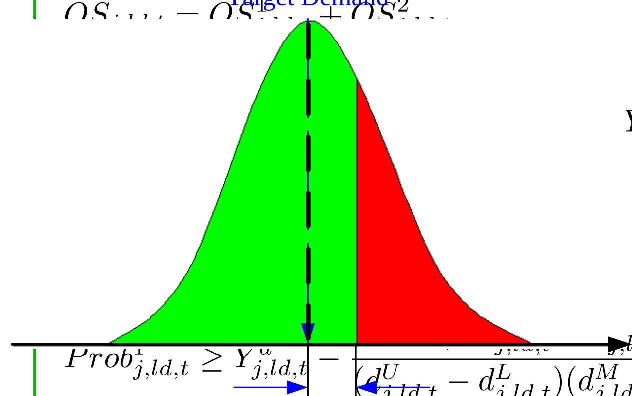
Chance constraint

$$\begin{bmatrix} y \\ d^L \leq QS \leq d^M \\ Prob = 1 - \dfrac{(QS - d^L)^2}{(d^U - d^L)(d^M - d^L)} \end{bmatrix} \vee \begin{bmatrix} \neg y \\ d^M \leq QS \leq d^U \\ Prob = \dfrac{(d^U - QS)^2}{(d^U - d^L)(d^U - d^M)} \end{bmatrix}$$

Generalized Disjunctive Programming



MINLP



$$QS_{j,ld,t} = QS^1_{j,ld,t} + QS^2_{j,ld,t}$$

$$Y^d_{j,ld,t})$$

$$Prob^1_{j,ld,t} \geq Y^d_{j,ld,t} - \frac{(\ldots_{j,ld,t})^2}{(d^U_{j,ld,t} - d^L_{j,ld,t})(d^M_{j,ld,t} - d^L_{j,ld,t})}$$

$$Prob^2_{j,ld,t} \geq \frac{[d^U_{j,ld,t}(1 - Y^d_{j,ld,t}) - QS^2_{j,ld,t}]^2}{(d^U_{j,ld,t} - d^L_{j,ld,t})(d^M_{j,ld,t} - d^L_{j,ld,t})}$$

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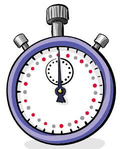
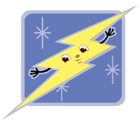
Objective Functions

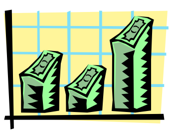
- Responsiveness
  - Measured by expected lead time





$$TP \geq Prob_{j,ld} Y^{I}_{k,ls} \lambda^{I}_{k,ls} + \sum_{x=1}^{n} Prob_{j,ld} Y^{P}_{k_x,i_x} + \sum_{x=1}^{n-1} Prob_{j,ld} Y^{N}_{k_x,i_{x+1}} \lambda^{N}_{k_x,i_x} + Prob_{j,ld} Y^{O}_{k_n,m} \lambda^{O}_{k_n,m} + Y^{S}_{m,ld} \lambda^{S}_{m,ld}$$

- Economics
  - Measured by net present value (NPV)



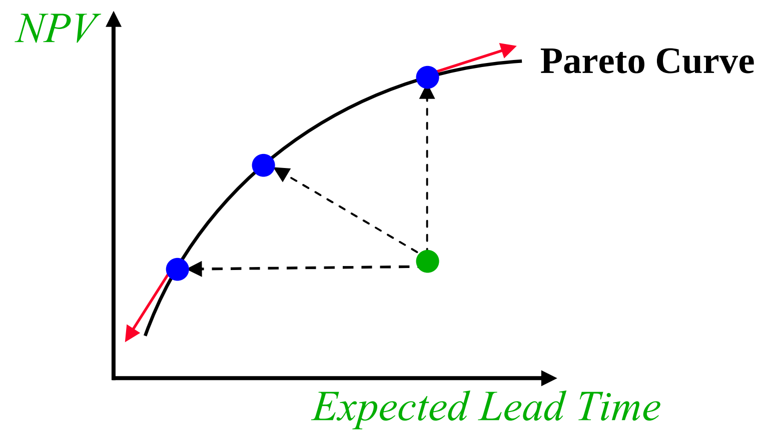
$$\begin{aligned}
NPV = & \boxed{\sum_{j}\sum_{ld}\sum_{t} Sales_{j,ld,t} \cdot Price_{j,ld,t}} && \rightarrow \text{Sales income} \\
& \boxed{-\sum_{k}\sum_{j}\sum_{ls}\sum_{t} Purch_{j,ls,t} \cdot RMCost_{k,j,ls,t}} && \rightarrow \text{Purchase cost} \\
& \boxed{-\sum_{k}\sum_{i}\sum_{s}\sum_{j \in JP_{i,s}}\sum_{t} prod_{k,i,j,s,t} \cdot UCost_{i,ls,t} - \sum_{k}\sum_{i}\sum_{t} SchCost_{k,i,t}} && \rightarrow \text{Operating cost} \\
& \boxed{-\sum_{k}\sum_{j}\sum_{ls}\sum_{t} tr_{k,j,ls,t} \cdot TCost_{k,j,ls,t} - \sum_{j}\sum_{m}\sum_{ld}\sum_{t} S_{j,m,ld,t} \cdot TCost_{j,m,ld,t}} && \rightarrow \text{Transport cost} \\
& \boxed{-\sum_{k}\sum_{i} Y_{k,i} \cdot IPCost_{k,i} - \sum_{m}\sum_{ld} Y_{m,ld} \cdot IDCost_{m,ld} - \sum_{k}\sum_{k'} Y_{k,k'} \cdot ILCost_{m,ld}} && \rightarrow \text{Investment cost} \\
& \boxed{-\sum_{j}\sum_{m}\sum_{ld}\sum_{t} Ss_{j,m,t} \cdot ICost_{j,m,ld,t} - \sum_{k}\sum_{j}\sum_{t} WI_{k,j,t} \cdot WICost_{k,j,t}} && \rightarrow \text{Inventory cost}
\end{aligned}$$

Bi-criterion Multiperiod MINLP Formulation

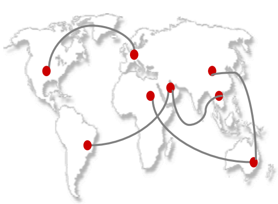
Choose Discrete (0-1), continuous variables

- Objective Function:
  - Max: Net Present Value
  - Min: Expected Lead time

  Bi-criterion

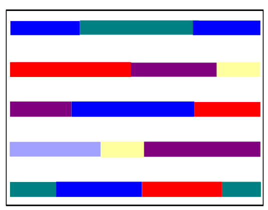


- Constraints:
  - Network structure constraints
    - Suppliers – plant sites Relationship
    - Plant sites – Distribution Center
    - Input and output relationship of a plant
    - Distribution Center – Customers
    - Cost constraint

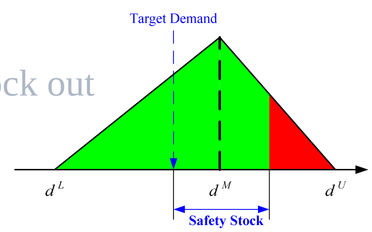


  - Operation planning constraints
    - Production constraint
    - Capacity constraint
    - Mass balance constraint
    - Demand constraint
    - Upper bound constraint

  - Cyclic scheduling constraints
    - Assignment constraint
    - Sequence constraint
    - Demand constraint
    - Production constraint
    - Cost constraint



  - Probabilistic constraints
    - Chance constraint for stock out (reformulations)



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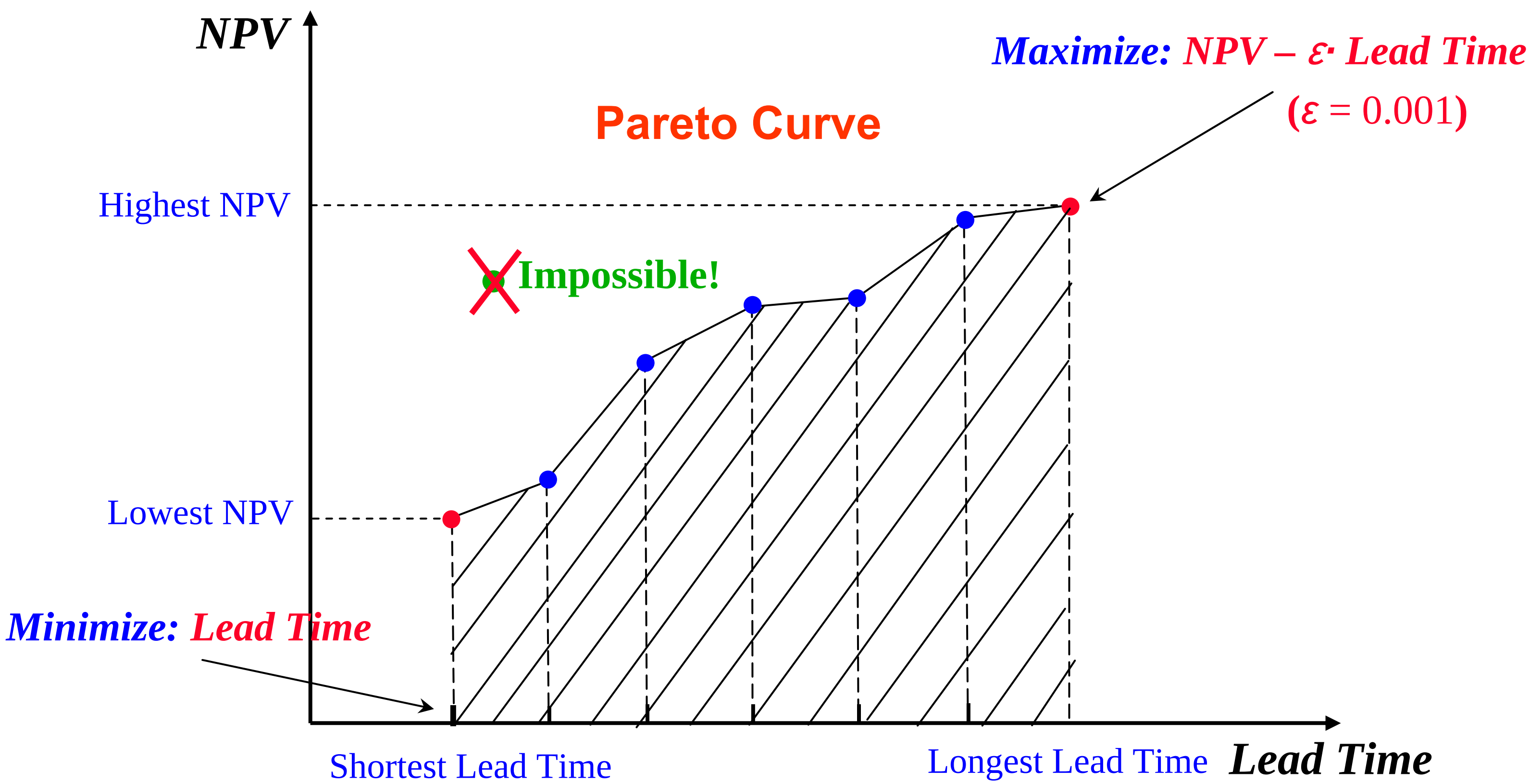
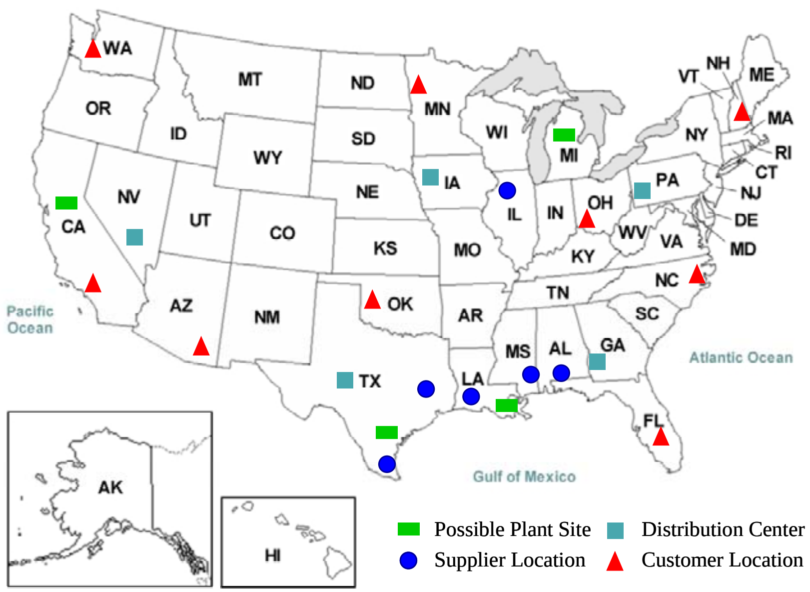
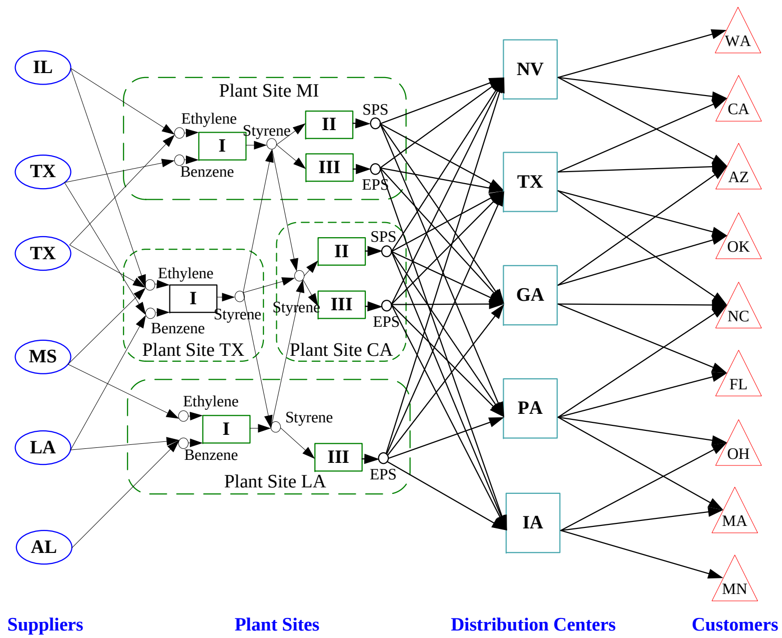
Responsive Supply Chains

Procedure for Pareto Optimal Curve



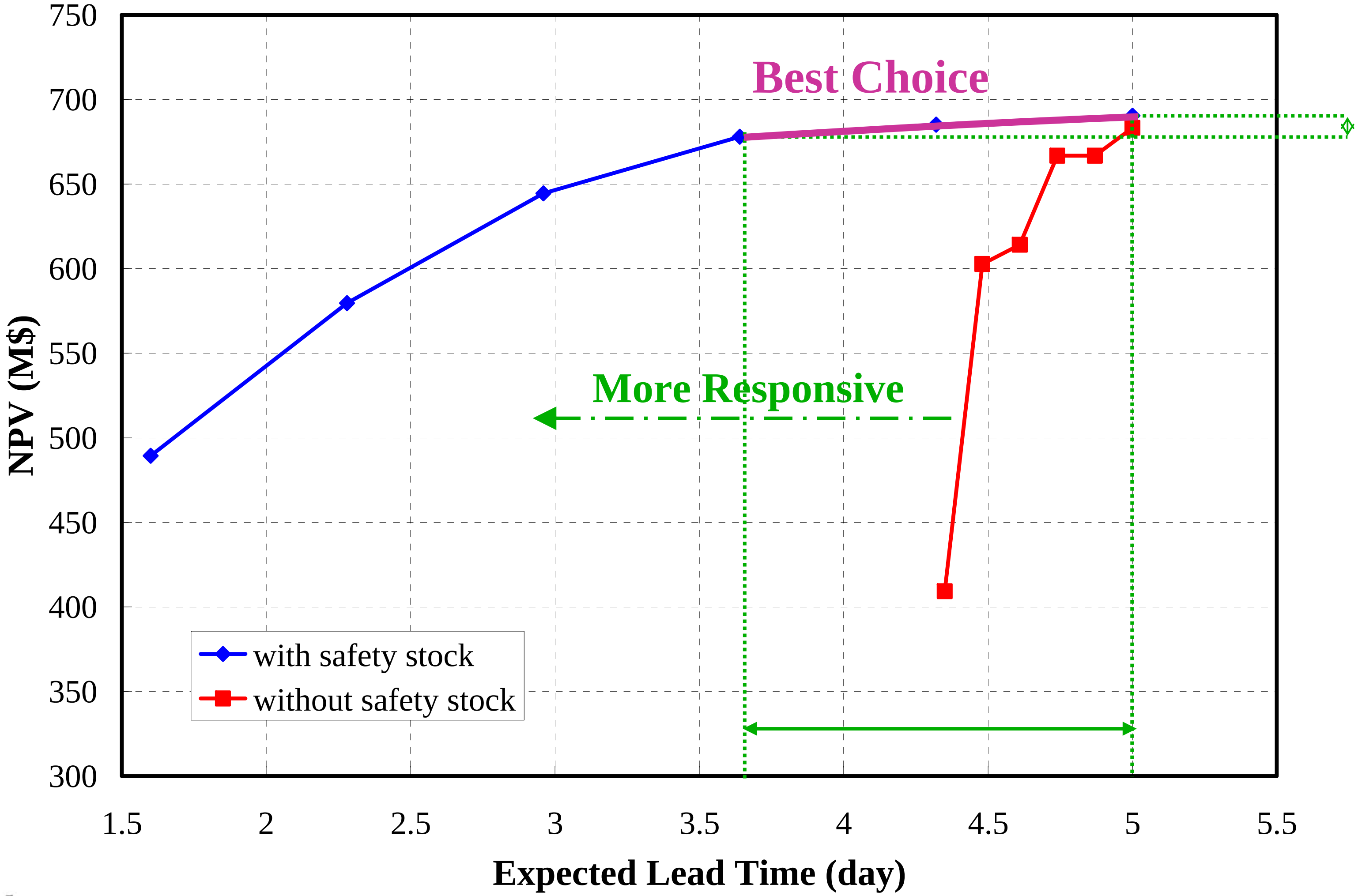
Case Study



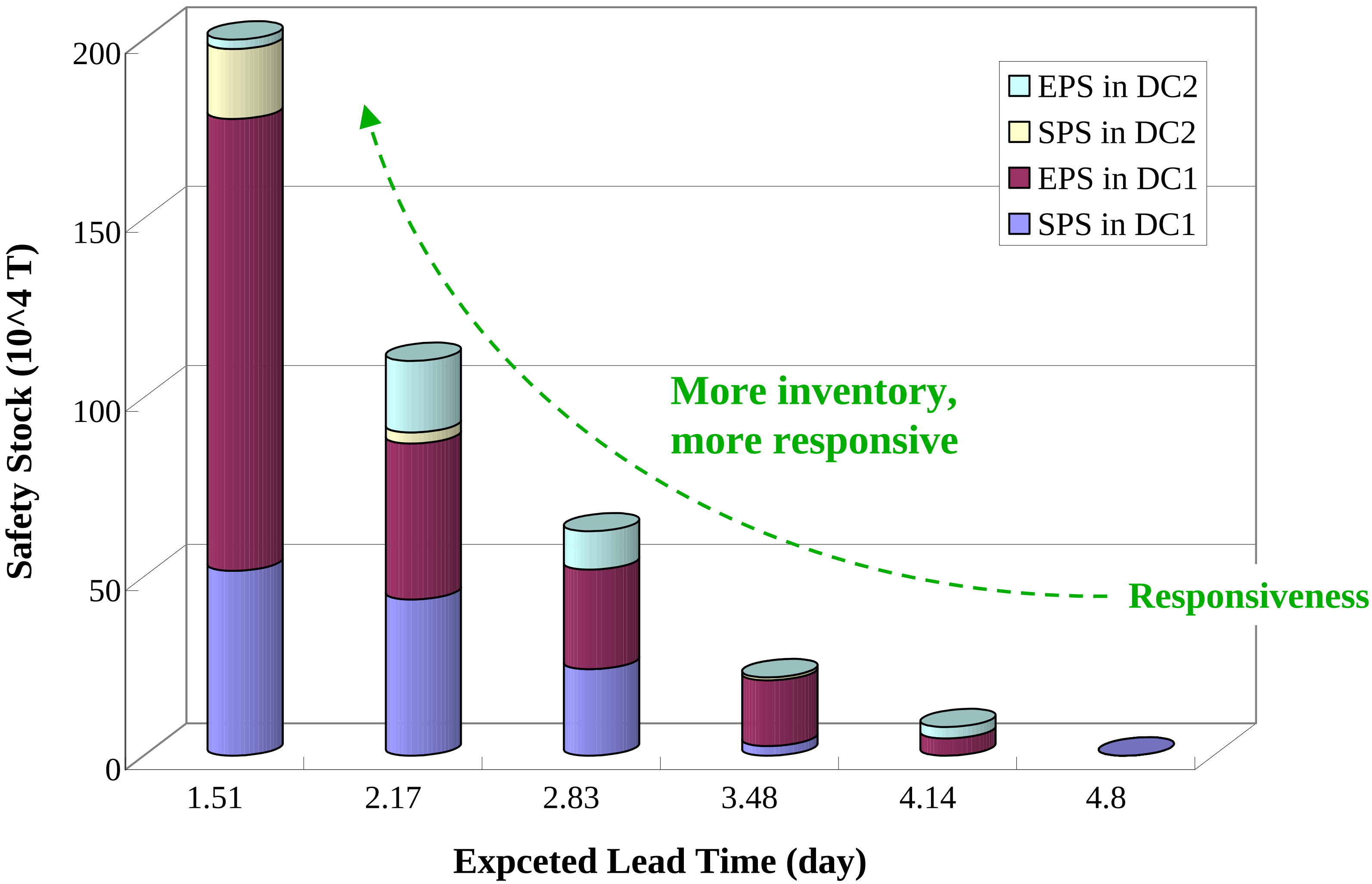


- Problem Size:
  - # of Discrete Variables: 215
  - # of Continuous Variables: 8126
  - # of Constraints: 14617
- Solution Time:
  - Solver: GAMS/BARON
  - Direct Solution: > 2 weeks
  - Proposed Algorithm: ~ 4 hours

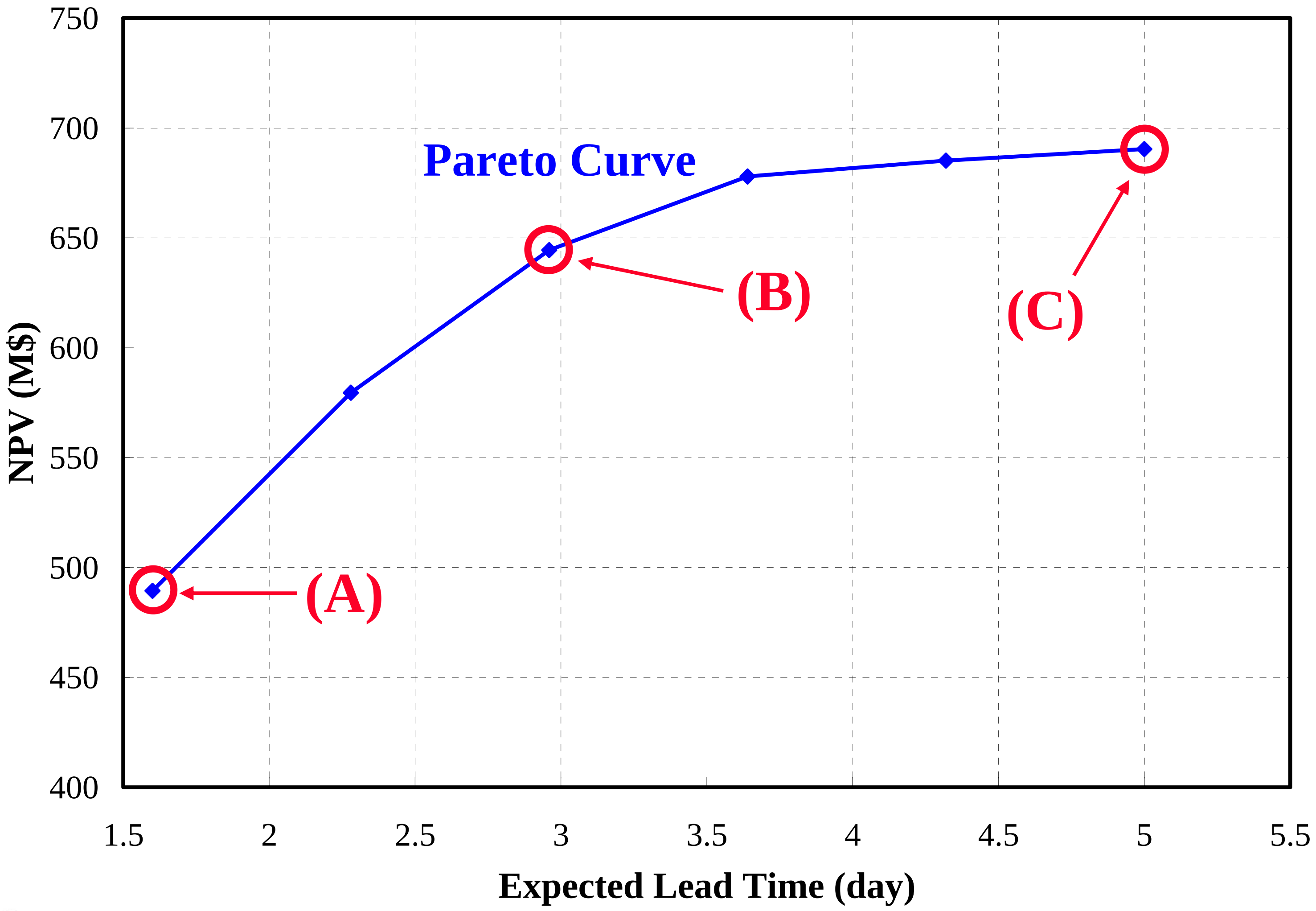
Pareto Curves – with and without safety stock



Safety Stock Levels - Expected Lead Time

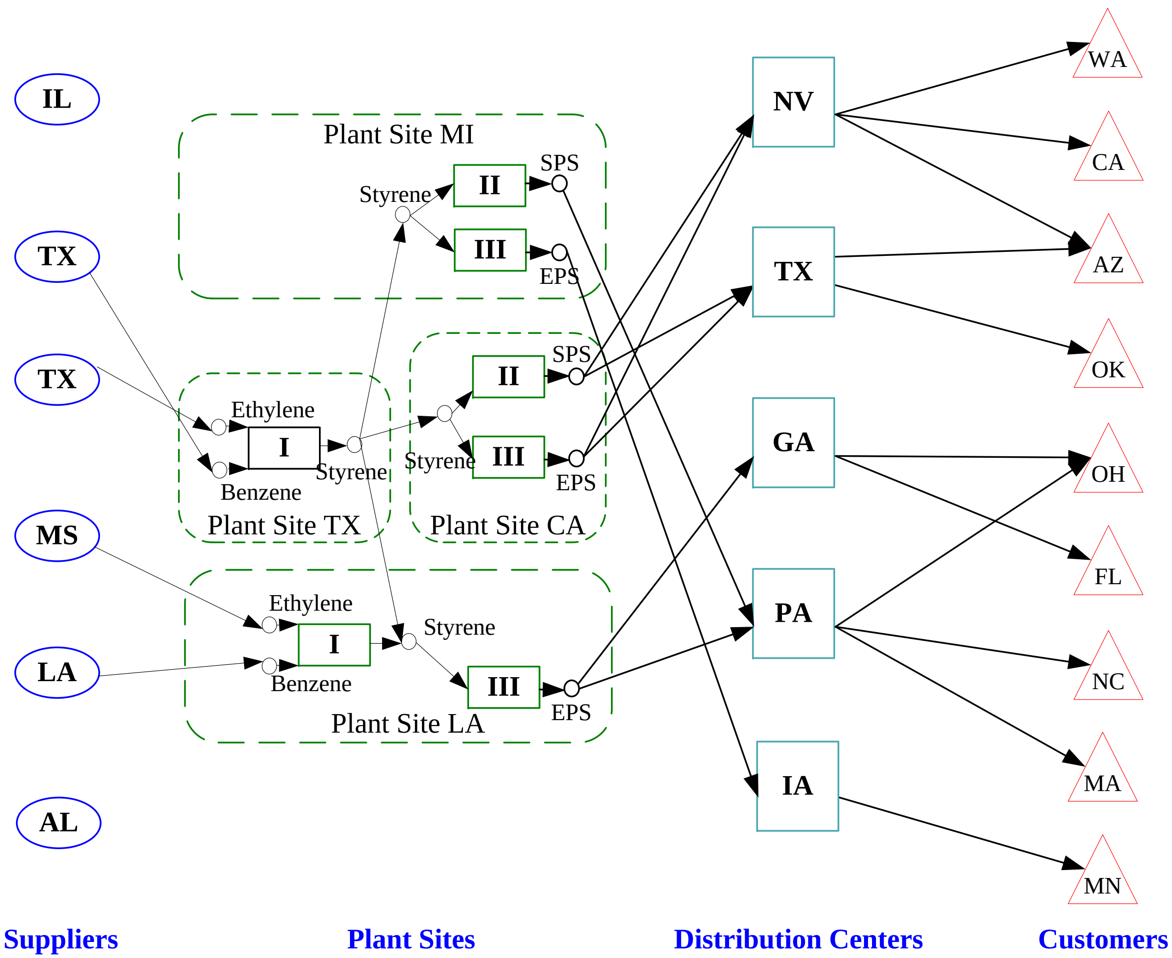


Optimal Network Structure



Optimal Network Structure – (A)

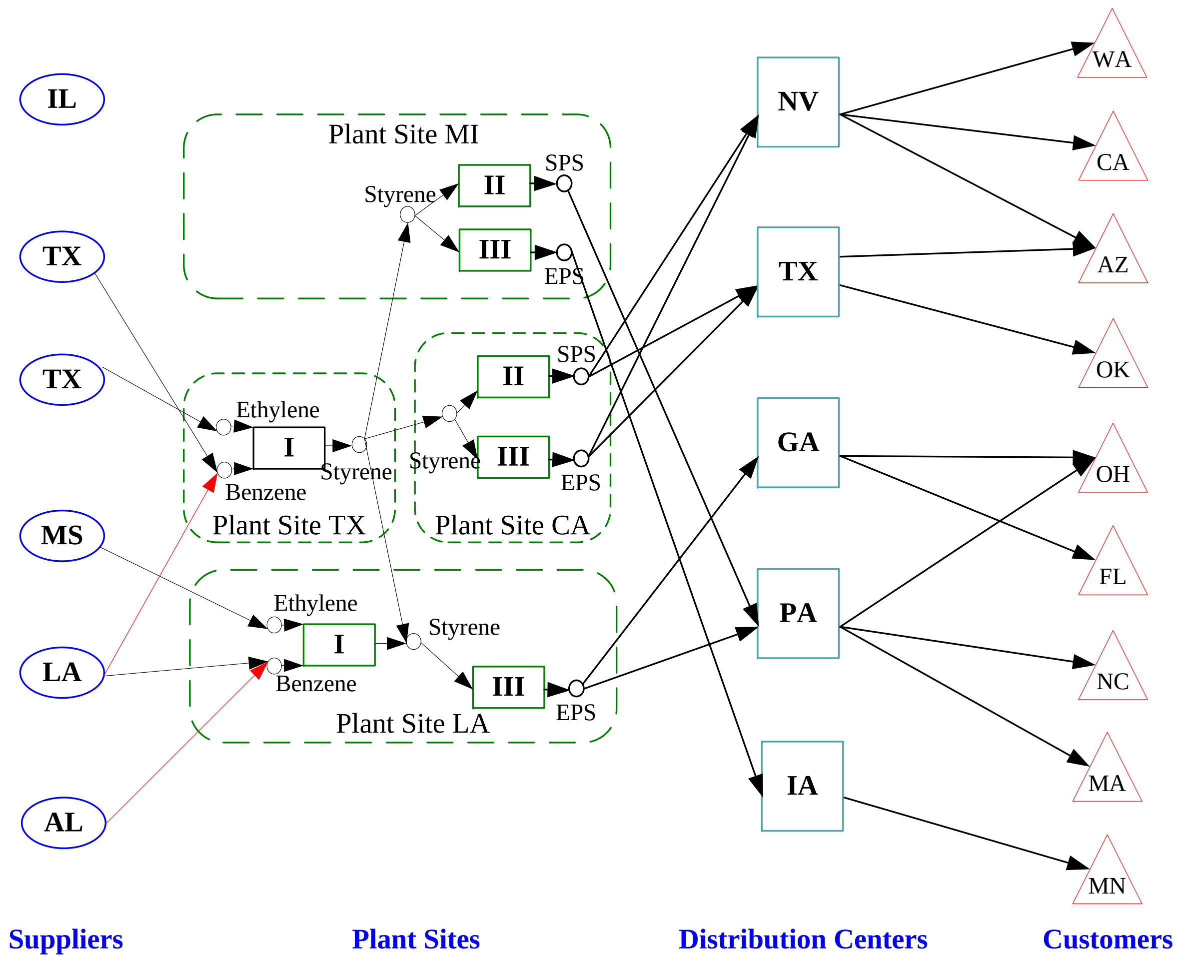
Shortest Expected Lead Time = 1.5 day   NPV = $489.39



Responsive Supply Chains

Optimal Network Structure – (B)

Expected Lead Time = 2.96 days   NPV = $644.46 MM



Responsive Supply Chains

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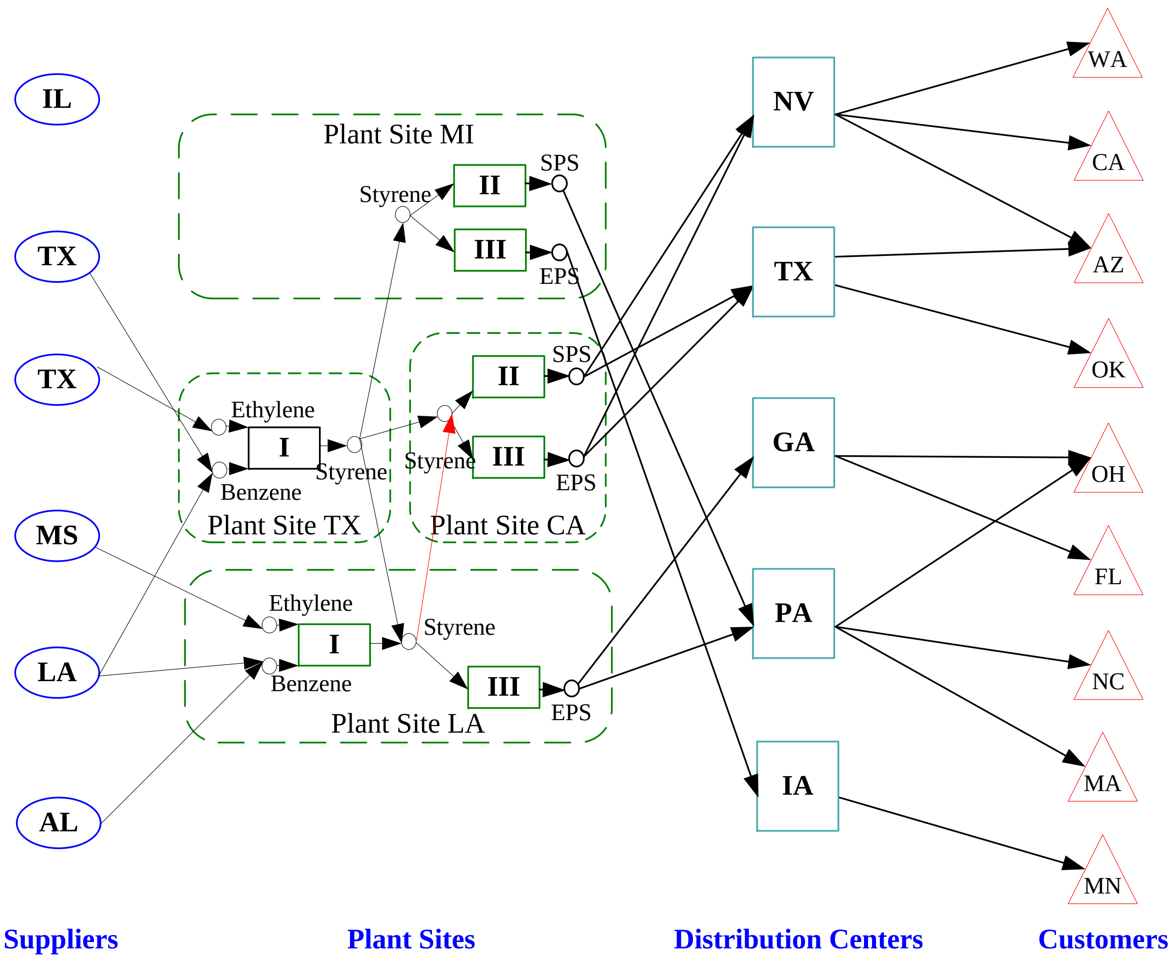
Optimal Network Structure – (C)

Longest Expected Lead Time = 5.0 day NPV = $690 MM



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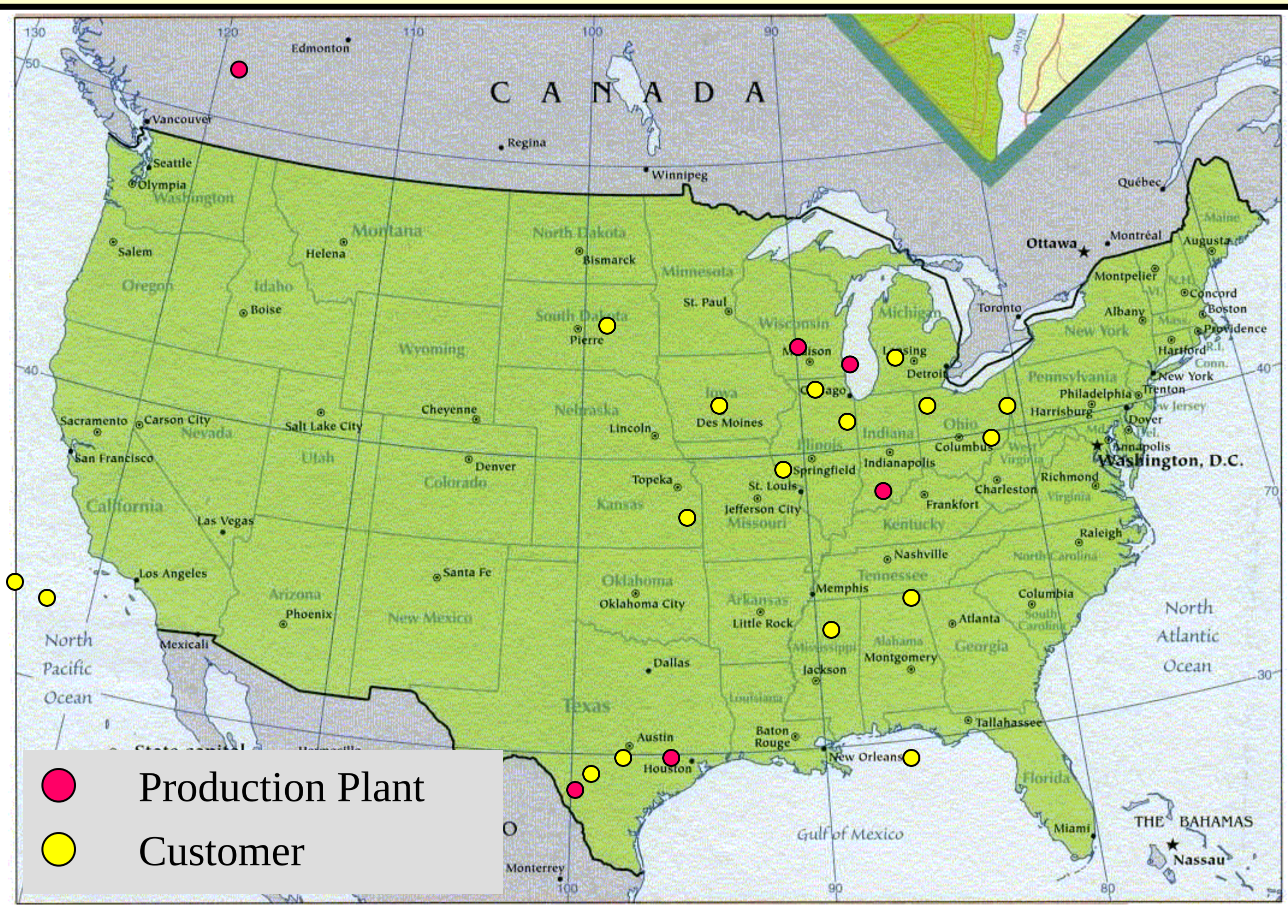
Simultaneous Tactical Planning and Production Scheduling

Goal: Improve the asset utilization of geographically distributed assets and reduce cost to serve by improving enterprise wide tactical production planning.



Multi-scale optimization: temporal and spatial integration

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Production Planning for Parallel Batch Reactors

Erdirik, Grossmann (2006)

Materials:

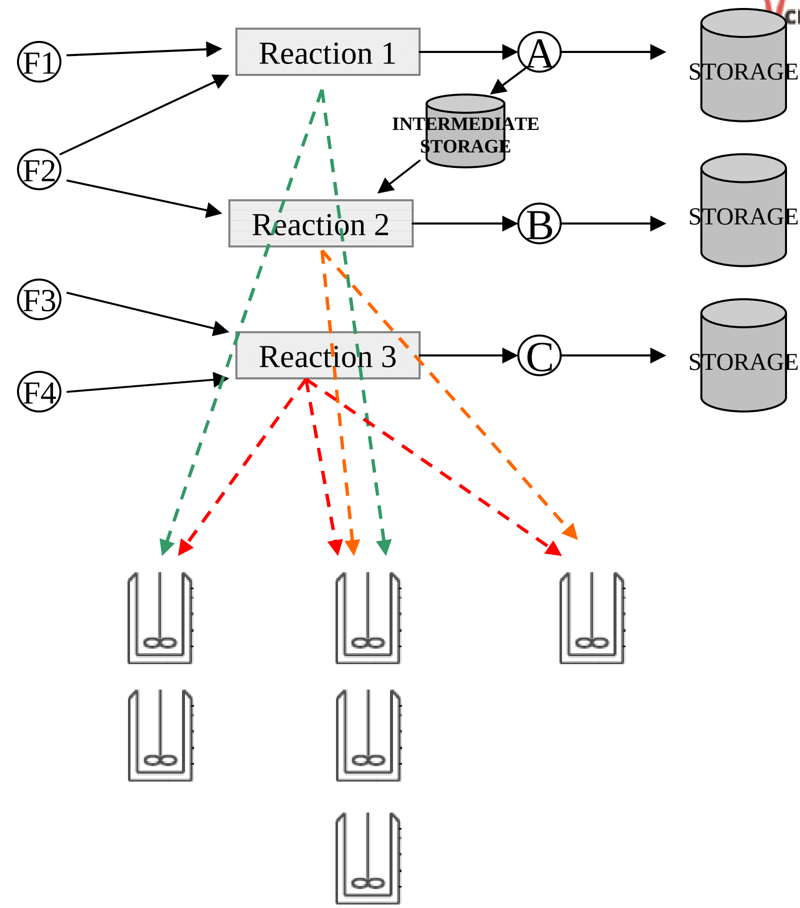
➢Raw materials, Intermediates, Finished products
➢Unit ratios (lbs of needed material per lb of material produced)

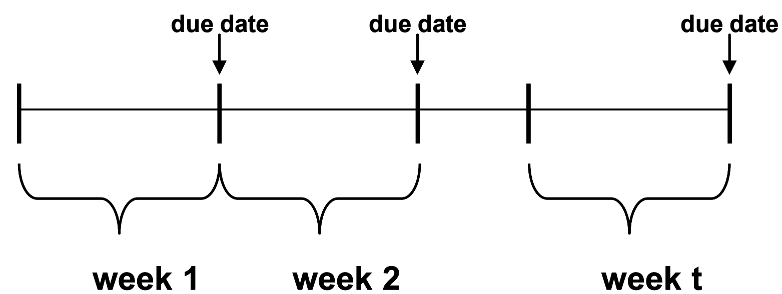
Production Site:

✓ Reactors:
- ✓ Products it can produce
- ✓ **Batch sizes** for each product
- ✓ **Batch process time** for each product (hr)
- ✓ Operating costs ($/hr) for each material
- ✓ ***Sequence dependent change-over times /costs***
- ✓ ***=> Lost capacity***
  (hrs per transition for each material pair)
- ✓ Time the reactor is available during a given month (hrs)

Customers:

➢Monthly forecasted demands for desired products
➢Price paid for each product





Problem Statement

DETERMINE THE PRODUCTION PLAN:

- ✓ Production quantities
- ✓ Inventory levels
- ✓ Number of batches of each product
- ✓ Assignments of products to available processing equipment
- ✓ ***Sequence of production*** in each processing equipment

OBJECTIVE:

To Maximize **Profit**.

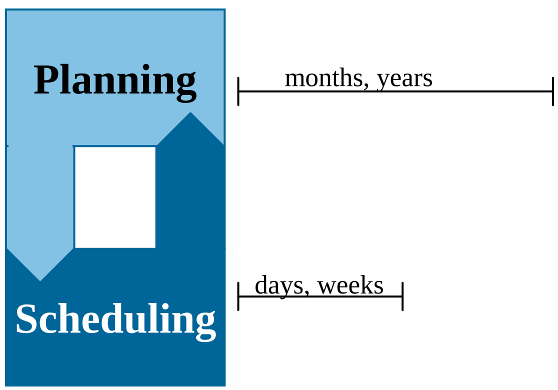
Profit = Sales – Costs

Costs=Operating Costs + Inventory Costs +Transition Costs

Approaches to Planning and Scheduling

Decomposition

Sequential Hierarchical Approach

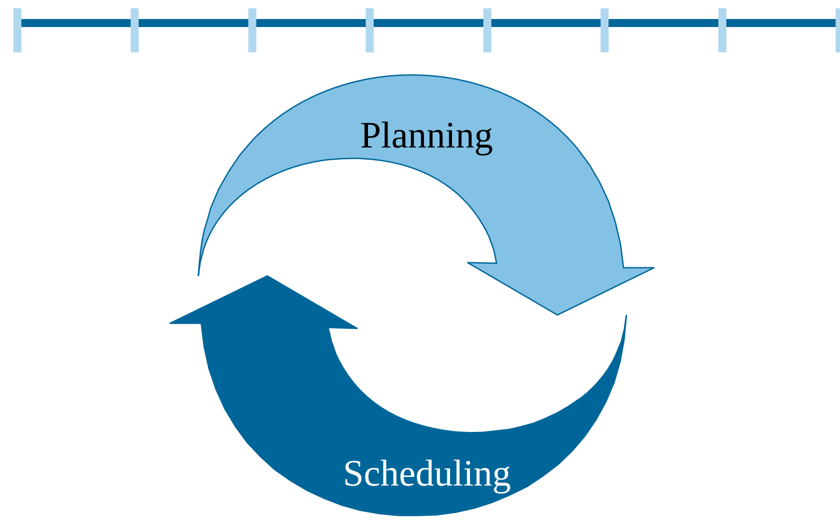


Challenges:

- ✓ Different models / different time scales
- ✓ **Mismatches between the levels**

Simultaneous Planning and Scheduling

Detailed scheduling over the entire horizon

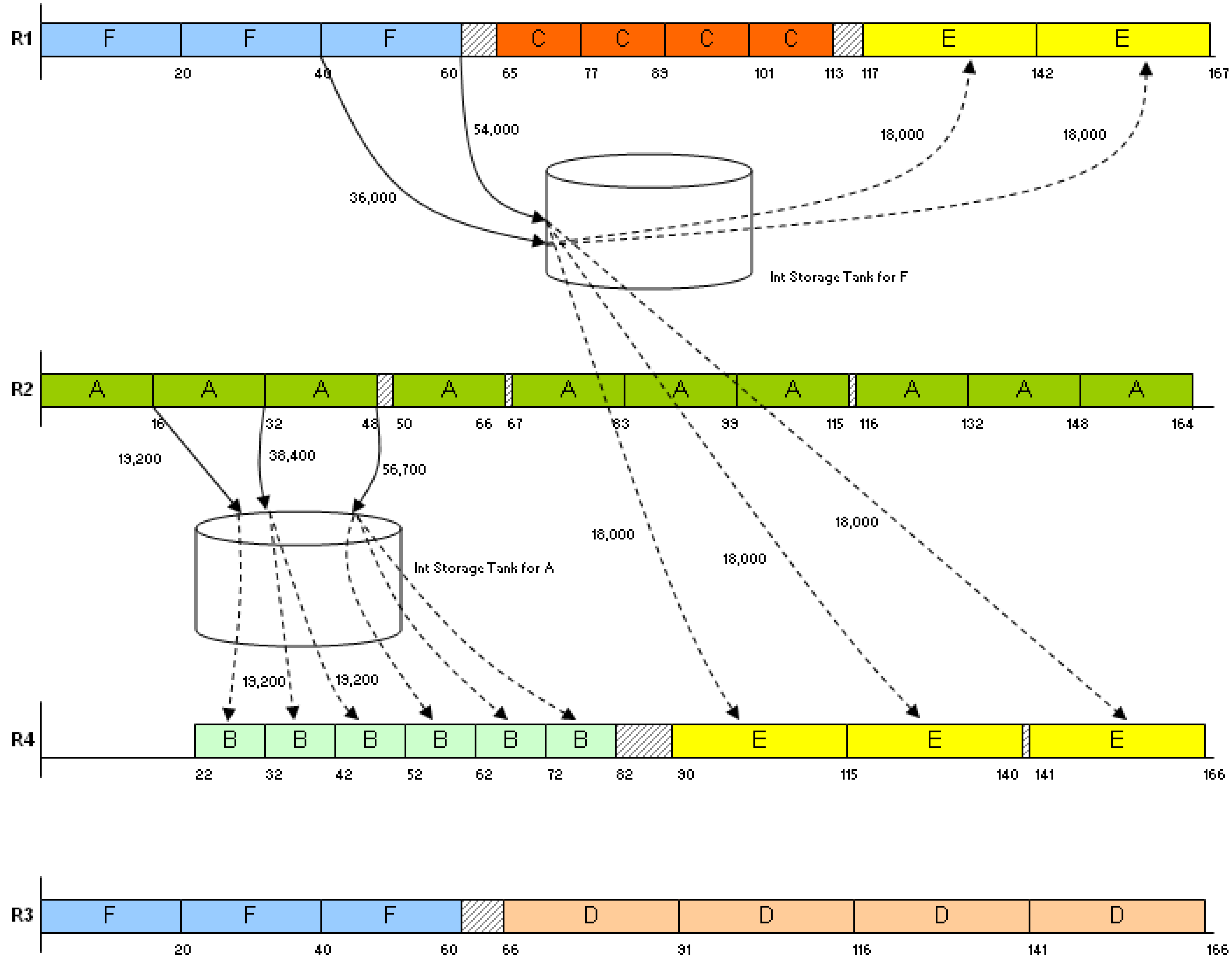


Challenges:

- ✓ Very Large Scale Problem
- ✓ **Solution times quickly intractable**

Goal: Planning model that integrates major aspects of scheduling

Results for Detailed MILP Scheduling Model: 4 reactors,6 products (1 week)



MILP Detailed Scheduling Model

Objective Function:

$$\Pr ofit = \sum_i \sum_t CP_{i,t} \cdot S_{i,t} - \sum_i \sum_m \sum_l \sum_t COP_{i,t} \cdot XB_{i,m,l,t} - \sum_i \sum_t CINV_{i,t} \cdot (INV_{i,t} + INVFIN_{i,t} + INVINT_{i,t}) - \sum_i \sum_k \sum_m \sum_l \sum_t CTRA_{i,k} \cdot \left(Z_{i,k,m,l,t} + \tilde{Z}_{i,k,m,l,t} + \hat{Z}_{i,k,m,l,t} \right)$$

Assignment constraints and Processing times:

$$\sum_i W_{i,m,l,t} \leq 1 \quad i \in IM(m), \forall\, l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$\sum_{i \in IM(m)} W_{i,m,l,t} \geq \sum_{i \in IM(m)} W_{i,m,l,t} \quad \forall l \in (L(m) \cap L(t)), l \neq N(t), \forall m, \forall t$$

$$PT_{i,m,l,t} = BT_{i,m} \cdot W_{i,m,l,t} \quad \forall i \in IM(m), \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$X_{i,m,l,t} = R_{i,m} \cdot BT_{i,m} \cdot W_{i,m,l,t} \quad \forall i \in IM(m), \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

Detailed timing constraints and sequence dependent change :

$$Z_{i,i',m,l,t} \geq W_{i,m,l,t} + W_{i',m,l+1,t} - 1 \quad \forall i \in IM(m), \forall i' \in IM(m), i' \neq i, \forall l \in (L(m) \cap L(t)), l \neq Nt, \forall m, \forall t$$

$$TR_{m,l,t} = \sum_i \sum_{i'} \tau_{i,i'} \cdot Z_{i,i',m,l,t} \quad \forall l \in (L(m) \cap L(t)), l \neq Nt, \forall m, \forall t$$

$$\tilde{Z}_{i,i',m,l,t} \geq W_{i,m,l,t} + W_{i',m,'1',t+1} - 1 \quad \forall i, i' \in IM(m), i' \neq i, \forall l \in (L(m) \cap L(t)), \forall m, \forall t, t \neq H_t$$

$$Te_{m,l,t} = Ts_{m,l,t} + \sum_i PT_{i,m,l,t} + \sum_i \sum_k \tau_{i,k} \cdot Z_{i,k,m,l,t} + TRT_{m,l,t} - TX_{m,l,t} + \left(\sum_i \sum_{i'} \tau_{i,i'} \cdot \hat{Z}_{i,i',m,l,t}\right) \quad \forall m, l, t$$

$$TRT_{m,l,t} = TRT1_{m,l,t} + TRT2_{m,l,t} \quad \forall m, l, t$$

$$TX_{m,l,t} = TRT1_{m,l,t} \quad \forall m, l, t$$

$$TRT1_{m,l,t} \leq UPPER \cdot Y_{m,l+1,t} \quad \forall m, l, t$$

$$TRT2_{m,l,t} \leq UPPER \cdot (1 - Y_{m,l+1,t}) \quad \forall m, l, t$$

MILP Detailed Scheduling Model

Mass and Inventory Balances:

$$X_{i,m,l,t} = INVP^{FIN}_{i,m,l,t} + INVINT^{TRA}_{i,m,l,t} \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$INVINT^{TRA}_{i,m,l,t} = INVP^{INT}_{i,m,l,t} + \sum_{l'>l, l' \in L(m)} AA_{i,m,l,m,l',t} + \sum_{m' \neq m} \sum_{l' \in L(m')} AA_{i,m,l,m',l',t} \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall l' \in L(t), \forall m, \forall m' \neq m, \forall t$$

$$X_{i,m,l,t} = INVP^{FIN}_{i,m,l,t} + INVP^{INT}_{i,m,l,t} + \sum_{l'>l, l' \in L(m)} AA_{i,m,l,m,l',t} + \sum_{m' \neq m} \sum_{l' \in L(m')} AA_{i,m,l,m',l',t} \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall l' \in L(t), \forall m, \forall m' \neq m, \forall t$$

$$Ts_{m,l,t} + \sum_{i} PT_{i,m,l,t} \leq Ts_{m',l',t} + BigW_t \cdot (1 - YY_{l,l',m,m',t}) \qquad \forall l \in (L(m) \cap L(t)), \forall l' \in (L(m') \cap L(t)), \forall m, \forall m' \neq m, \forall t$$

$$Ts_{m',l',t} \leq Ts_{m,l,t} + \sum_{i} PT_{i,m,l,t} + BigW_t \cdot (YY_{l,l',m,m',t}) \qquad \forall l \in (L(m) \cap L(t)), \forall l' \in (L(m') \cap L(t)), \forall m, \forall m' \neq m, \forall t$$

$$AA_{i,m,l,m',l',t} \leq UBOUND_i \cdot (YY_{l,l',m,m',t}) \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall l' \in (L(m') \cap L(t)), \forall m, \forall m' \neq m, \forall t$$

$$\sum_{m' \neq m} \sum_{l' \in (L(m') \cap L(t))} AA_{i,m,l,m',l',t} \leq UBOUND_i \cdot (W_{i,m,l,t}) \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$\sum_{l' \in (L(m) \cap L(t))} AA_{i,m,l,m',l',t} \leq UBOUND_i \cdot (W_{i,m,l,t}) \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$AA_{i,m,l,m',l',t} \leq UPBOUND_{i,m} \cdot \sum_{i' \in ENDINT(i',i)} (W_{i',m,l,m',l',t}) \qquad i \in IFINT_i, \forall l \in (L(m) \cap L(t)), \forall l' \in (L(m') \cap L(t)), \forall m, \forall m' \neq m, \forall t$$

$$INV^{INT}_{i,t-1} + \sum_{m} \sum_{l \in (L(m) \cap L(t))} INVP^{INT}_{i,m,l,t} = \sum_{m} \sum_{l \in (L(m) \cap L(t))} INVC_{i,m,l,t} + INV^{INT}_{i,t} \qquad i \in IFINT_i, \forall t$$

$$INV^{FIN}_{i,t-1} + \sum_{m} \sum_{l \in (L(m) \cap L(t))} INVP^{FIN}_{i,m,l,t} = S_{i,t} + INV^{FIN}_{i,t} \qquad i \in IFINT_i, \forall t$$

$$X_{i,m,l,t} = \sum_{i'} \alpha_{i,i'} \cdot \Big(INVC_{i',m,l,t} + \sum_{\substack{l''<l \\ l' \in (L(m) \cap L(t))}} AA_{i,m,l'',m,l,t} + \sum_{m' \neq m} \sum_{l' \in (L(m') \cap L(t))} AA_{i,m',l',m,l,t}\Big) \qquad i \in IE_i, \forall l \in (L(m) \cap L(t)), \forall m, \forall t$$

$$INV^{FIN}_{i,t-1} + \sum_{m} \sum_{l \in (L(m) \cap L(t))} X_{i,m,l,t} = S_{i,t} + INV^{FIN}_{i,t} \qquad i \in (IE_i \cup IF_i), \forall t$$

Proposed MILP Planning Models

Replace the detailed timing constraints by:

Model A. (Relaxed Planning Model)

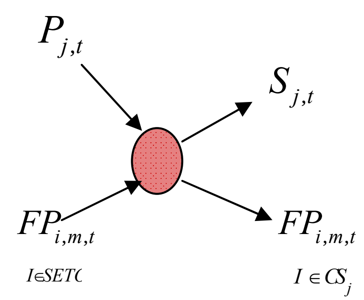
- ✓ Constraints that **underestimate the sequence dependent** changeover times
- ✓ **Weak** upper bounds **(Optimistic Profit)**

Model B. (Detailed Planning Model)

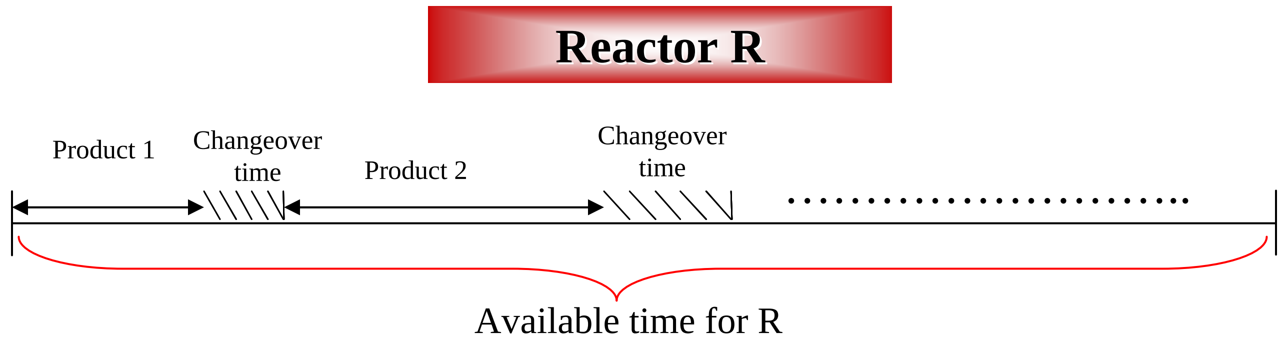
- ✓ **Sequencing constraints** for accounting for transitions rigorously
  (Traveling salesman constraints)
- ✓ **Tight** upper bounds **(Realistic estimate Profit)**

Generic Form of Proposed MILP Planning Models

✓ Mass Balances on State Nodes



✓ Time Balance Constraints on Equipment



✓ Objective Function

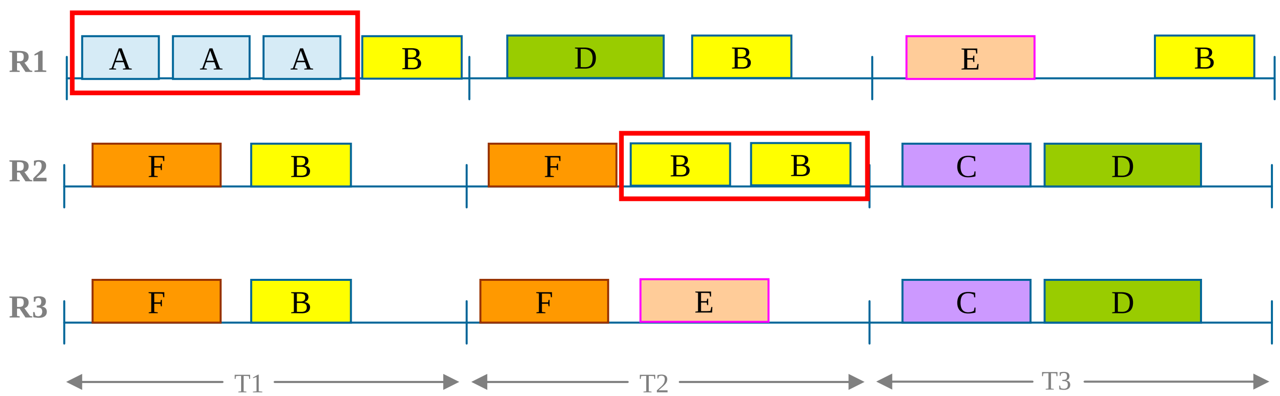
Key Variables for Model

$YP_{i,m,t}$:the assignment of products to units at each time period

NB_{imt} :number of each batches of each product on each unit at each period

FP_{imt} :amount of material processed by each task

Products: A, B, C, D, E, F → Reactor 1 or Reactor 2 or Reactor 3



$$YP_{A,reactor1,time1} = 1$$
$$NB_{A,reactor1,time1} = 3$$

$$YP_{B,reactor2,time2} = 1$$
$$NB_{B,reactor2,time2} = 2$$

Proposed Model B (Detailed Planning)

Sequence dependent changeovers (traveling salesman constraints):

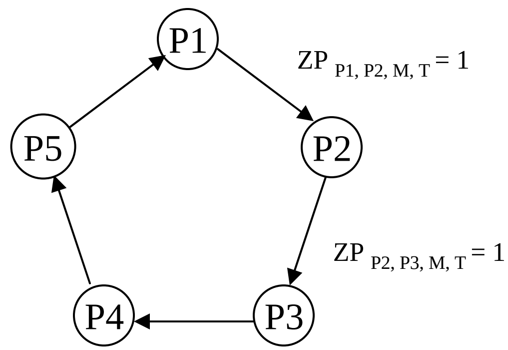
✓ Changeovers within each time period:

1. **Generate a cyclic schedule where total transition time is minimized.**

KEY VARIABLE:

$ZP_{ii'mt}$:becomes 1 if product i is after product i' on unit m at time period t, zero otherwise

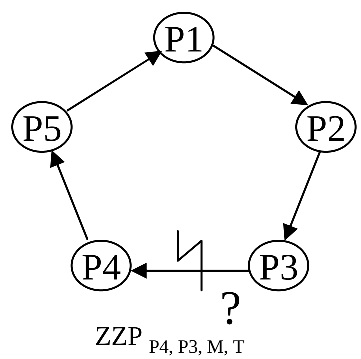
P1, P2, P3, P4, P5



2. **Break the cycle at the pair with the maximum transition time to obtain the sequence.**

KEY VARIABLE:

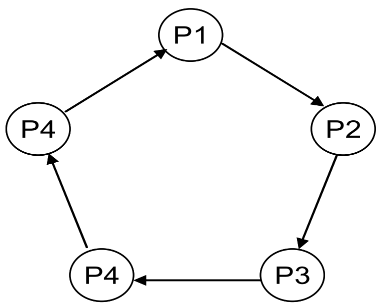
$ZZP_{ii'mt}$:becomes 1 if the link between products i and i' is to be broken, zero otherwise



=> P4→P5→P1→P2→P3

MILP Model

According to the location of the link to be broken:



P2, P3, P4, P5, P1 $\longrightarrow$ $ZZP_{P1,\,P2,\,M,\,T} = 1$

P3, P4, P5, P1, P2 $\longrightarrow$ $ZZP_{P2,\,P3,\,M,\,T} = 1$

P4, P5, P1, P2, P3 $\longrightarrow$ $ZZP_{P3,\,P4,\,M,\,T} = 1$

P5, P1, P2, P3, P4 $\longrightarrow$ $ZZP_{P4,\,P5,\,M,\,T} = 1$

P1, P2, P3, P4, P5 $\longrightarrow$ $ZZP_{P5,\,P1,\,M,\,T} = 1$

The sequence with the minimum total transition time is the **optimal sequence** within time period t.

$$YP_{imt} = \sum_{i'} ZP_{ii'mt} \qquad \forall i,m,t$$

$$YP_{i'mt} = \sum_{i} ZP_{ii'mt} \qquad \forall i',m,t$$

$$YP_{imt} \wedge \left[\bigwedge_{i'\neq i} \neg YP_{i'mt}\right] \Leftrightarrow ZP_{iimt} \qquad \forall i,m,t$$

$$YP_{imt} \geq ZP_{i,i,m,t} \qquad \forall i,m,t$$

$$ZP_{i,i,m,t} + YP_{i',m,t} \leq 1 \qquad \forall i,i' \neq i,m,t$$

$$ZP_{i,i,m,t} \geq YP_{i,m,t} - \sum_{i'\neq i} YP_{i',m,t} \qquad \forall i,m,t$$

$$\sum_{i}\sum_{i'} ZZP_{ii'mt} = 1 \qquad \forall m,t$$

$$ZZP_{ii'mt} \leq ZP_{ii'mt} \qquad \forall i,i',m,t$$

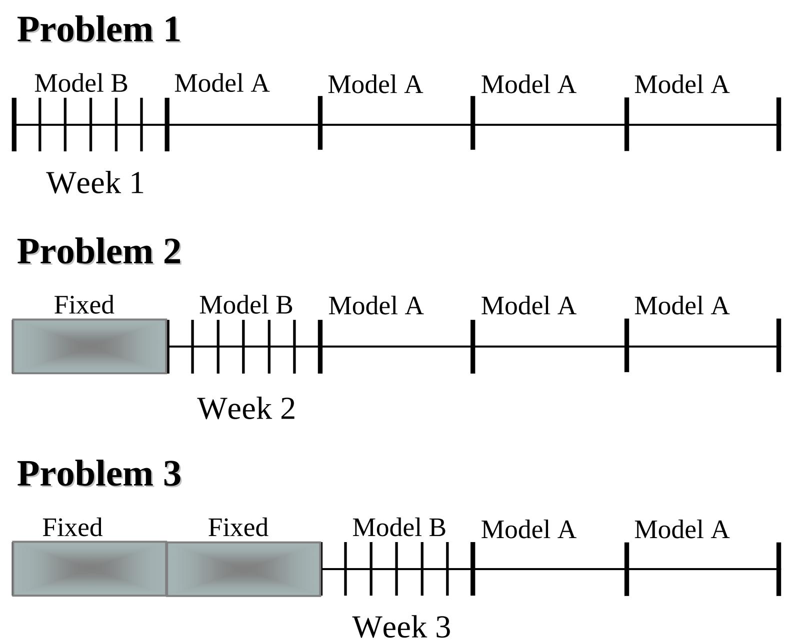
Generate the cycle and break the cycle to find the optimum sequence where transition times are minimized.

Having determining the sequence, we can determine **the total transition time** within each week.

Limitation: Large Problems

The **proposed planning models** may be expensive to solve for long term horizons.

ROLLING HORIZON APPROACH :



- ✓ The detailed planning period (Model B) moves as the model is solved in time.
- ✓ Future planning periods include only underestimations for transition times.

*Ref. Dimitriadis et al, 1997

EXAMPLE: 5 Products, 2 Reactors, 1 Week

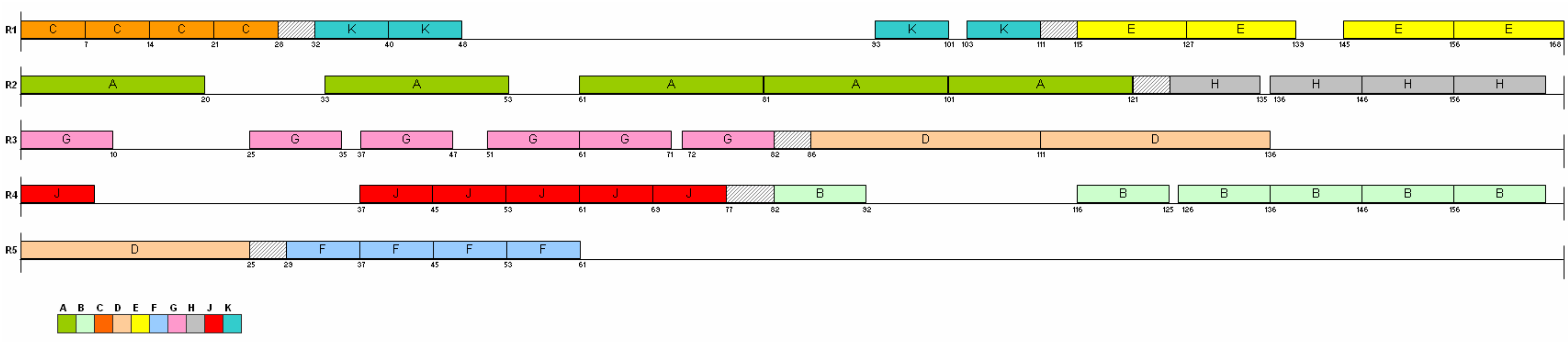
	Method	Number of binary variables	Number of continuous variables	Number of Equations	Time (CPUs)	Solution ($)
A	Relaxed Planning	20	49	67	0.046	**1,680,960.0**
B	Detailed Planning	140	207	335	0.296	**1,571,960.0**
	Scheduling	594	2961	2537	150	**1,571,960.0**

% 6.484 Difference

Obj Function Items ($)	Relaxed Planning	Detailed Planning	Scheduling
Sales	2,652,800	2,440,000	2,440,000
Operating Costs	971,840	868,000	868,000
Transition Costs	0	40	40
Inventory Costs	0	0	0

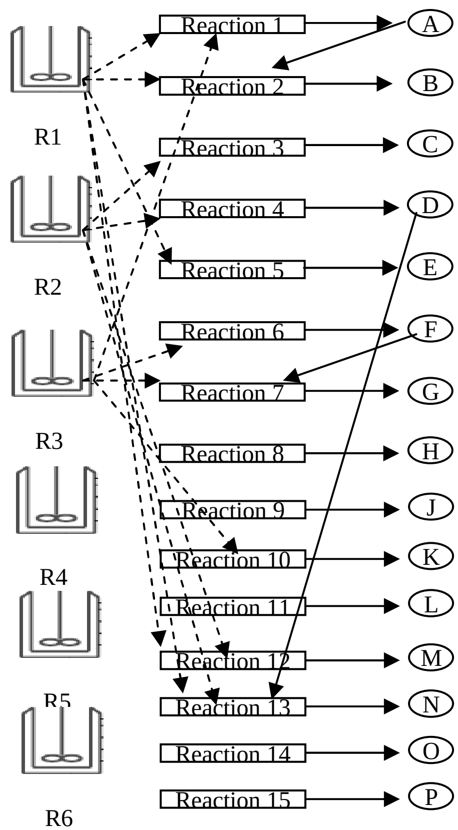
Detailed Planning and Scheduling are Identical!

Gantt Chart:



EXAMPLE 2 - 15 Products, 6 Reactors, 48 Weeks

✓ Determine the plan for 15 products, 6 reactors plant so as to maximize profit.



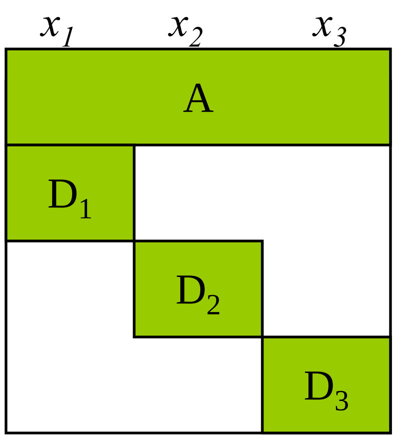
- 15 Products, A,B,C,D,E,F,G,H,J,K,L,M,N,O,P
- B, G and N are produced in 2 stages.
- 6 Reactors, R1,R2,R3,R4,R5,R6
- End time of the week is defined as due dates
- Demands are lower bounds

Relaxed planning yields 21% overestimation of profit

method	number of binary variables	number of continuous variables	number of equations	time (CPU s)	solution ($)
relaxed planning (A)	2,592	5,905	9,361	362	224,731,683
rolling horizon (RH)	10,092	25,798	28,171	11,656	184,765,965

Decomposable MILP Problems

Complicating Constraints

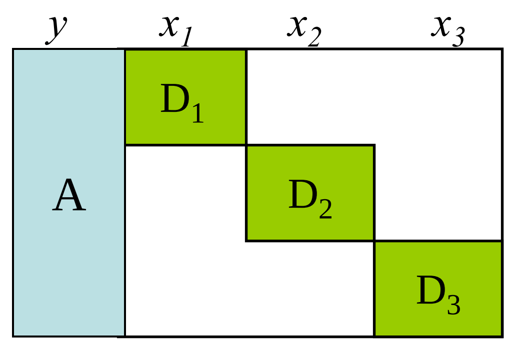


complicating constraints

$$\max\ c^T x$$
$$st\ \ Ax = b$$
$$D_i x_i = d_i \ \ i = 1,..n$$
$$x \in X = \{x \mid x_i\,, i = 1,..n, \mid x_i \geq 0\}$$

Lagrangean decomposition

Complicating Variables



complicating variables

$$\max\ a^T y + \sum_{i=1,..n} c_i^T x_i$$
$$st\ \ Ay + D_i x_i = d_i \ \ i = 1,..n$$
$$y \geq 0,\ x_i \geq 0, i = 1,..n$$

Benders decomposition

Note: can reformulate by defining

$y_i = y_{i+1}$ **Complicating constraints**

and apply Lagrangean decomposition

Lagrangean Relaxation *(Fisher, 1985)*

- **MILP optimization problems can often be modeled as problems with complicating constraints.**

- **The complicating constraints are added to the objective function (i.e. dualized) with a penalty term (Lagrangean multiplier) proportional to the amount of violation of the dualized constraints.**

- **The Lagrangean problem is easier to solve *(eg. can be decomposed)* than the original problem and provides an upper bound to a maximization problem.**

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Lagrangean Relaxation

$$
\begin{aligned}
Z = \max \quad & cx \\
s.t.\ & Ax \le b \\
& Dx \le e \\
& x \in Z_+^n
\end{aligned}
\qquad \textbf{(IP)}
$$

Assume integers only
Easily extended cont. vars.

Assume that $Ax \le b$ **is complicating constraint**

$$
\begin{aligned}
Z_{LR}(u) = \max \quad & cx + u(b - Ax) \\
& Dx \le e \\
& x \in Z_+^n
\end{aligned}
$$

$$where \qquad u \ge 0 \quad Lagrange\ multipliers$$

Carnegie Mellon

Lagrangean Relaxation

$$Z = \max \quad cx$$

Complicating Constraint $\longrightarrow$ $Ax \le b$

$$Dx \le e$$

$$x \in Z_+^n$$

$\Rightarrow$

$$Z_{LR}(u) = \max \quad cx + u(b - Ax)$$

$$Dx \le e$$

$$x \in Z_+^n$$

$$where \qquad u \ge 0$$

This is a relaxation of original problem because:

i) removing the constraint $Ax \le b$ **relaxes the original feasible space,**

ii) $Z_{LR}(u) \ge Z$ **always holds as in the original space since** $(b - Ax) \ge 0$ **and Lagrange multiplier is always** $u \ge 0$**.**

Lagrangean Relaxation Yields Upper Bound $\Rightarrow \quad Z_{LR}(u) \ge \quad Z$

Lagrangean Relaxation

Original problem:

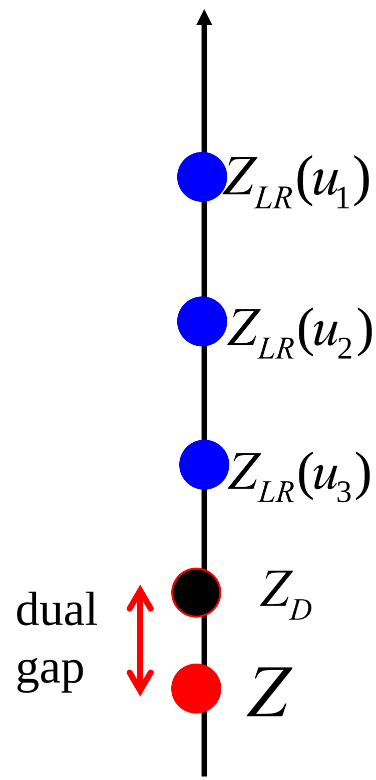
$$Z = \max \quad cx$$
$$s.t.\ Ax \le b$$
$$Dx \le e$$
$$x \in Z_+^n$$

Relaxed problem:

$$Z_{LR}(u) = \max \quad cx + u(b - Ax)$$
$$Dx \le e$$
$$x \in Z_+^n$$

Lagrangean dual:

$$Z_D = \min Z_{LR}(u)$$
$$u \ge 0$$



Graphical Interpretation

Relaxed problem:

$$\boxed{\begin{aligned} Z_{LR}(u) = \max \quad & cx + u(b - Ax) \\ & Dx \le e \\ & x \in Z_+^n \end{aligned}}$$

Lagrangean dual:

$$Z_D = \min Z_{LR}(u)$$
$$u \ge 0$$

Combine Relaxed and Lagrangean Dual Problems:

$$Z_D = \min_{u \ge 0} \left\{ \max_{x \ge 0} \quad cx + u(b - Ax) \right\}$$
$$Dx \le e$$
$$x \in Z_+^n$$

Graphical Interpretation

$$Z_D = \min_{u \geq 0} \left\{ \max_{x \geq 0} \quad cx + u(b - Ax) \right\}$$
$$Dx \leq e$$
$$x \in Z_+^n$$

$\Longleftrightarrow$

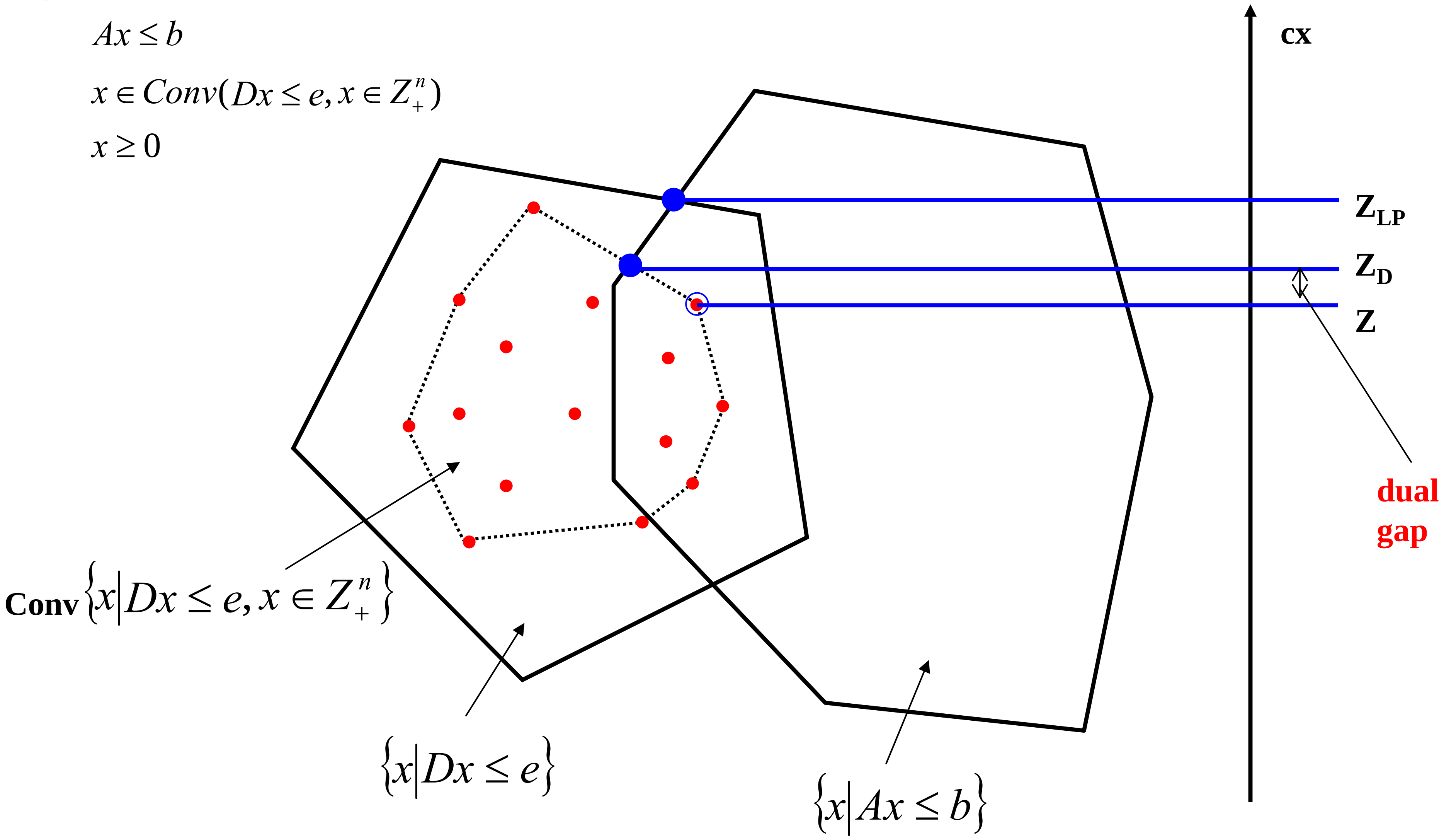
Nice Proof
Frangioni (2005)

$$Z_D' = \max cx$$
$$Ax \leq b$$
$$x \in Conv(Dx \leq e, x \in Z_+^n)$$
$$x \geq 0$$

Optimization of Lagrange multipliers (dual) can be interpreted as optimizing the primal objective function on the intersection of the convex hull of non-*complicating* constraints set $\{x | Dx \leq e, x \in Z_+^n\}$ and the LP relaxation of the *relaxed* constraints set $\{x | Ax \leq b, x \in Z_+^n\}$.

Graphical Interpretation

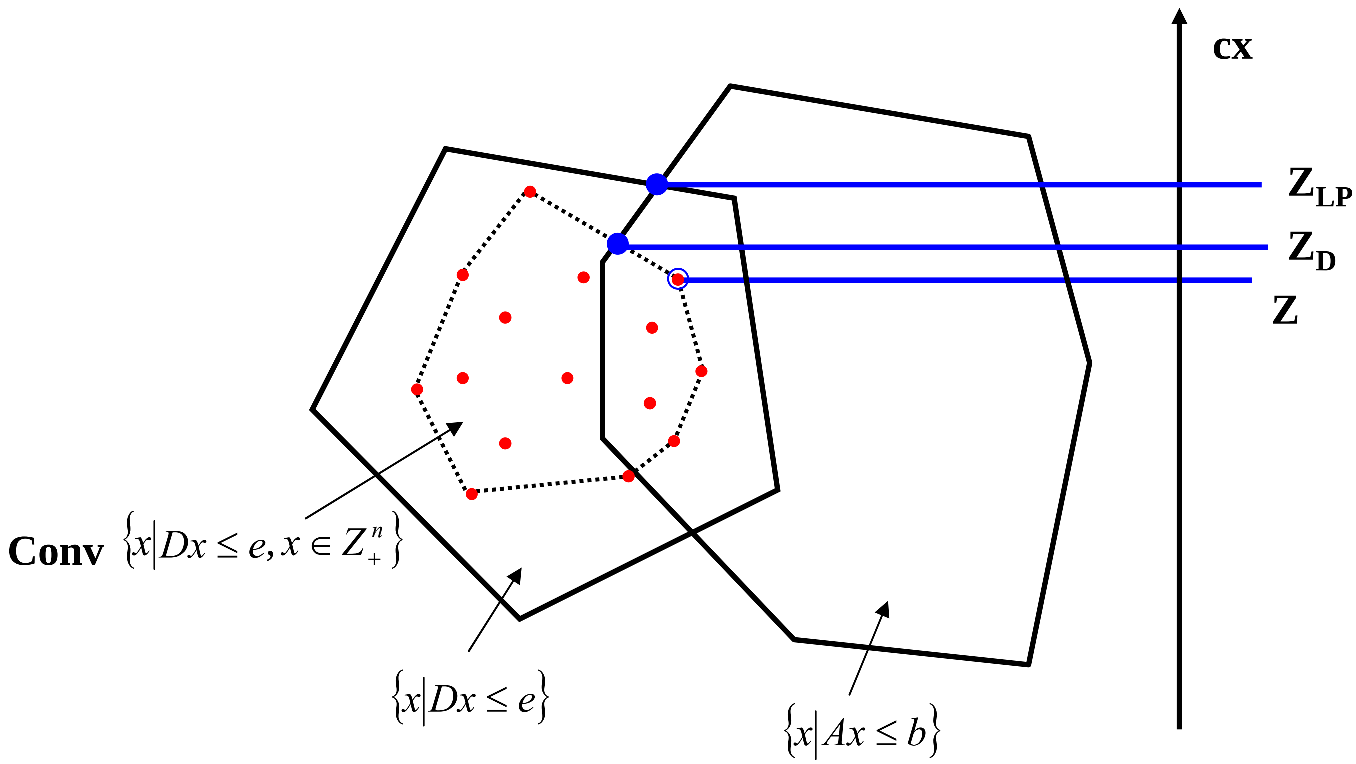
$$Z_D' = \max cx$$
$$Ax \le b$$
$$x \in Conv(Dx \le e, x \in Z_+^n)$$
$$x \ge 0$$



Theorem

Lagrangean relaxation yields a bound at least as tight as LP relaxation

$$Z(P) \leq Z_D \leq Z_{LR}(u) \leq Z_{LP}$$



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Lagrangean Decomposition *(Guignard & Kim, 1987)*

- **Lagrangean Decomposition is a special case of Lagrangean Relaxation.**
- **Define variables for each set of constrain, add constraints equating different variables *(new complicating constraints)* to the objective function with some penalty terms.**

$$\begin{aligned} Z = \max \quad & cx \\ s.t.\ & Ax \le b \\ & \boxed{Dx \le e} \\ & x \in Z_+^n \end{aligned}$$

$$\begin{aligned} Z' = \max \quad & cx \\ & Ax \le b \\ & \boxed{Dy \le e} \\ & x = y \\ & x \in Z_+^n \\ & y \in Z_+^n \end{aligned}$$

New complicating constraints

$$\begin{aligned} Z_{LD}(v) = \max \quad & cx + v(y - x) \\ & Ax \le b \\ & Dy \le e \\ & x \in Z_+^n \\ & y \in Z_+^n \end{aligned}$$

Dualize *x* = *y*

Lagrangean Decomposition

$$Z_{LD}(v) = \max \quad cx + v(y - x)$$
$$Ax \le b$$
$$Dy \le e$$
$$x \in Z_+^n$$
$$y \in Z_+^n$$

Subproblem 1

$$Z_{LD1}(v) = \max \quad (c - v)x$$
$$Ax \le b$$
$$x \in Z_+^n$$

Subproblem 2

$$Z_{LD2}(v) = \max \quad vy$$
$$Dy \le e$$
$$y \in Z_+^n$$

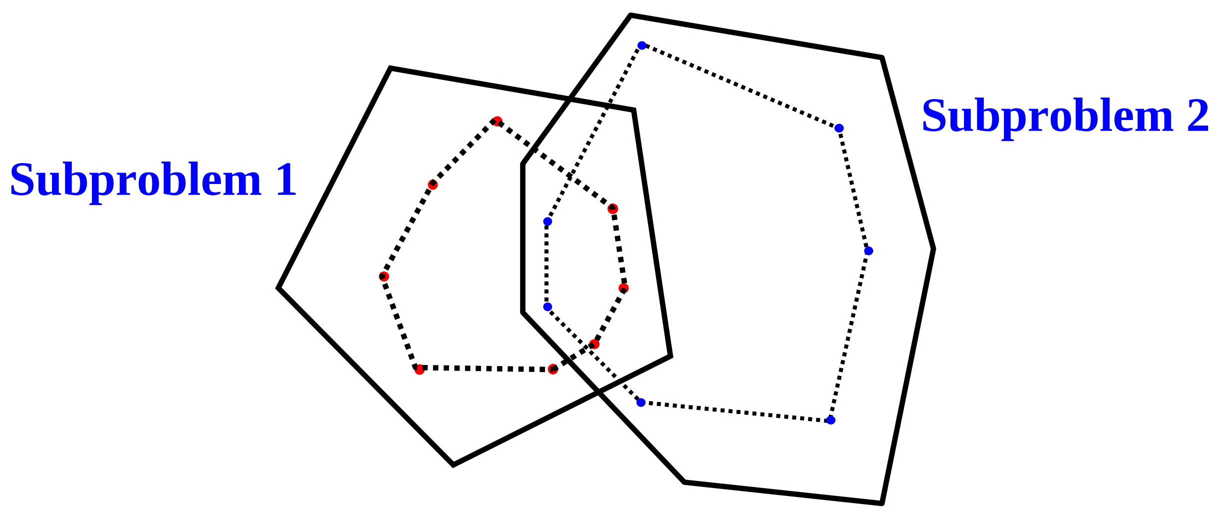
$$Z_{LD} = \min_{v \ge 0} \quad \left(Z_{LD1}(v) + Z_{LD2}(v) \right)$$

Lagrangean dual

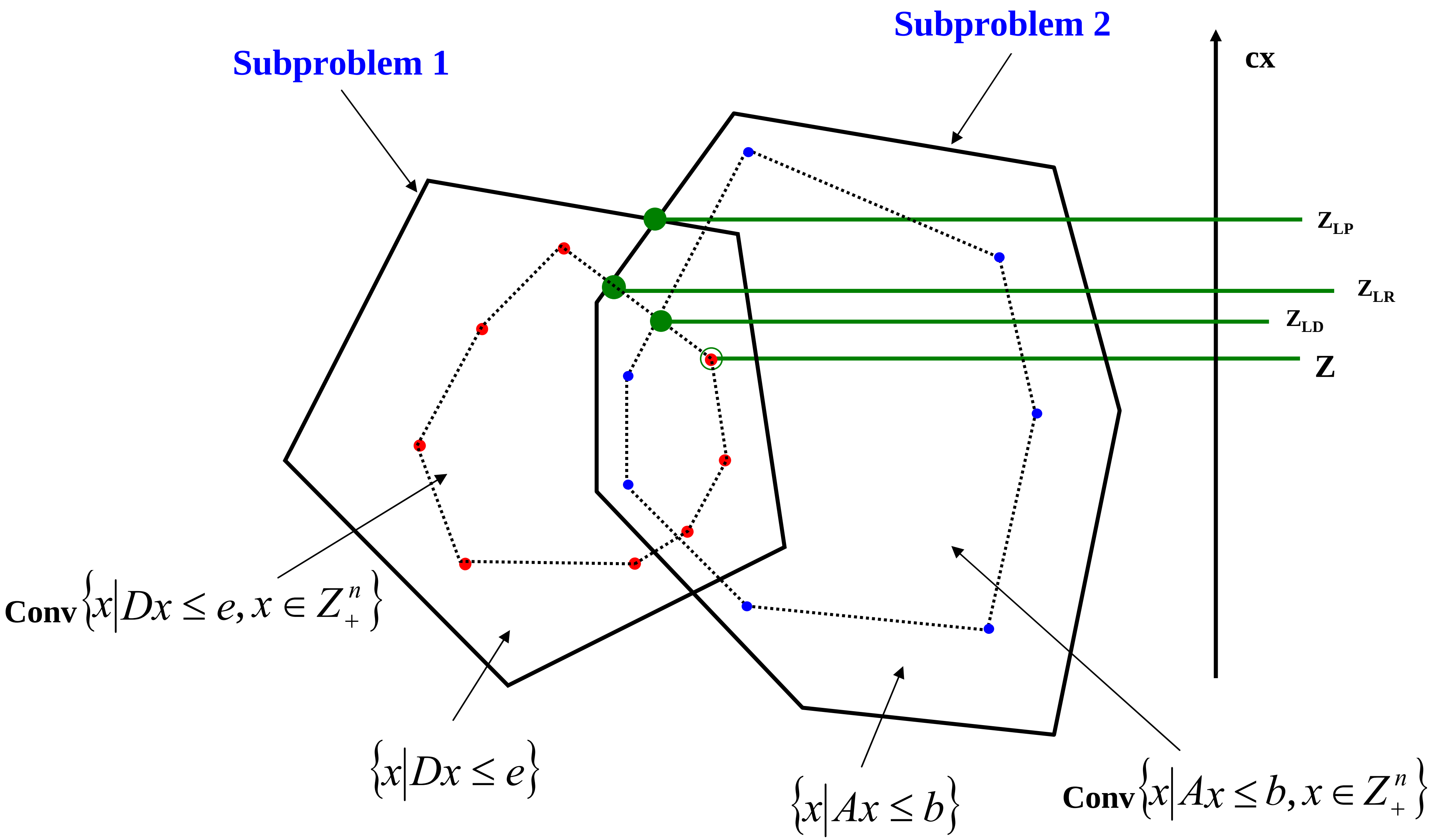
Notes

➢ Lagrangean decomposition is different from other possible relaxations because every constraint in the original problem appears in one of the subproblems.

<u>Graphically:</u> The optimization of Lagrangean multipliers can be interpreted as optimizing the primal objective function on the intersection of the convex hulls of constraint sets.



Graphical Interpretation?



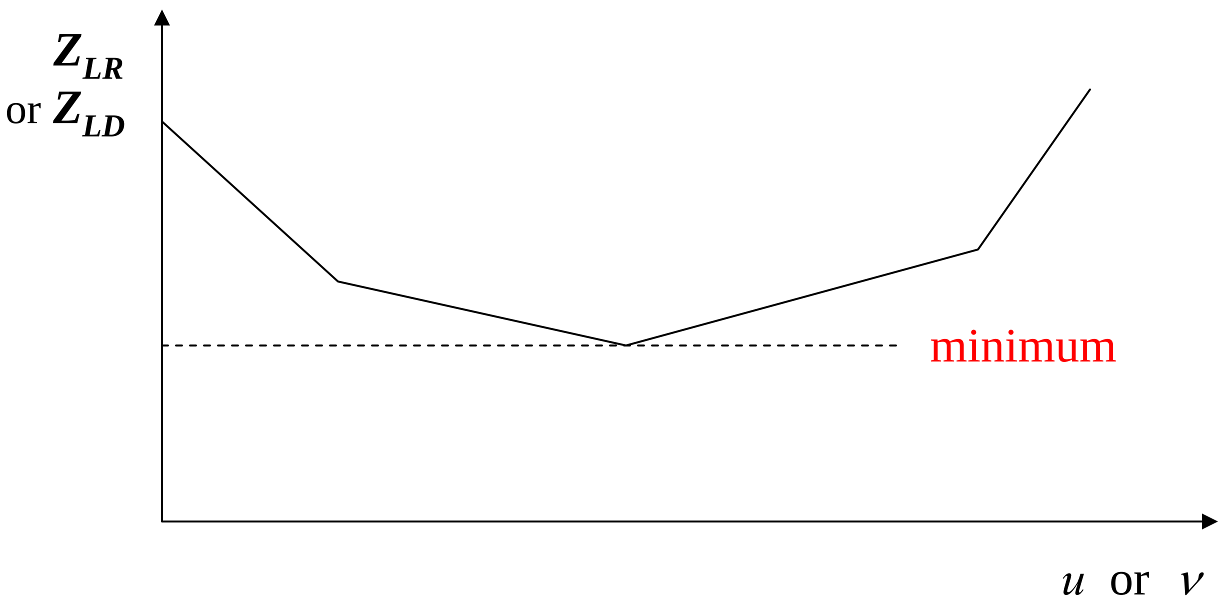
Note: Z_{LR}, Z_{LD} refer to dual solutions

Theorem

- **The** **bound** **predicted by “Lagrangean decomposition” is at least as tight as the one provided by “Lagrangean relaxation”** ***(Guignard and Kim, 1987)***
- **For a maximization problem**

$$Z(P) \leq Z_{LD} \leq Z_{LR} \leq Z_{LP}$$

Solution of Dual Problem



Piecewise linear

⇓

Non-differentiable

How to iterate on multipliers u?

Assuming $Dx \le d$ is a bounded polyhedron (polytope) with extreme points

$x^k \; k = 1, 2 \ldots K$, **then**

$$\max_x \{ cx + u(b - Ax) \,|\, Dx \le d, x \in X \} = \max_{k=1,\ldots K} \{ cx^k + u(b - Ax^k) \}$$

$\Downarrow$ **Dual problem**

$$\min_{u \ge 0} \max_{k=1,..K} \{ cx^k + u(b - Ax^k) \} = \min_{u \ge 0} \{ \eta \,|\, \eta \ge cx^k + u(b - Ax^k), k = 1,..K \}$$

Cutting plane approach

$$\min \; \eta$$

subgradient

$$s.t. \; \eta \ge cx^k + u(b - Ax^k), \; k = 1,..K_n$$

$$u \ge 0, \; \eta \in R^1$$

K_n = no. extreme points iteration n

Note: x^k generated from *max{cx + u^k(b-Ax)* subproblems

Subgradient Optimization Approach

Subgradient $s^k = (b - Ax^k)$

Steepest descent search $u^{k+1} = u^k + \mu s^k$

Update formula for multipliers *(Fisher, 1985)*

$$u^{k+1} = u^k + \alpha_k (Z^{LB} - Z_{LD}{}^k)(b - Ax^k) / \left\| b - Ax^k \right\|^2$$

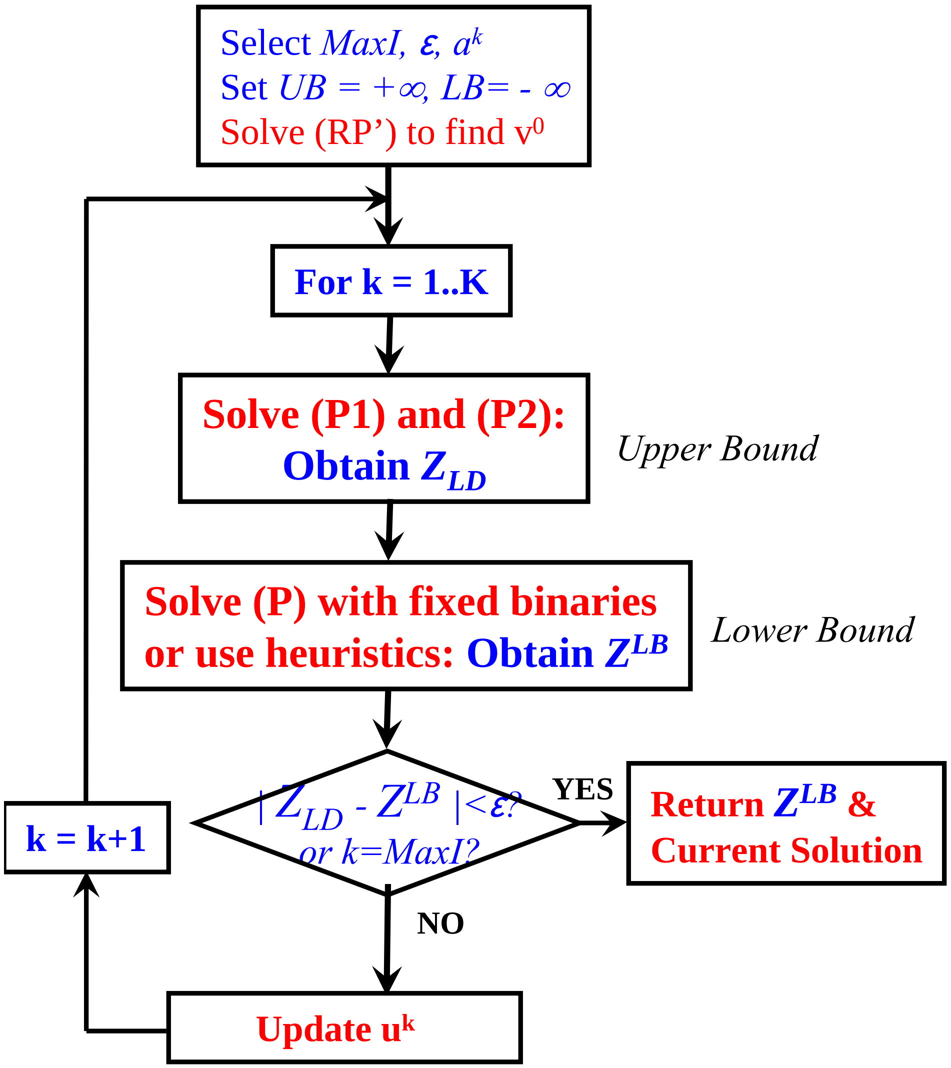
$$where \ \alpha_k \in [0, 2]$$

Note: Can also use bundle methods for nondifferentiable optimization

Lemarechal, Nemirovski, Nesterov (1995)

Solution of Langrangean Decomposition

1. Iterative search in multilpliers of dual



Notes: Heuristic due to dual gap
Obtaining Lower Bound might be tricky

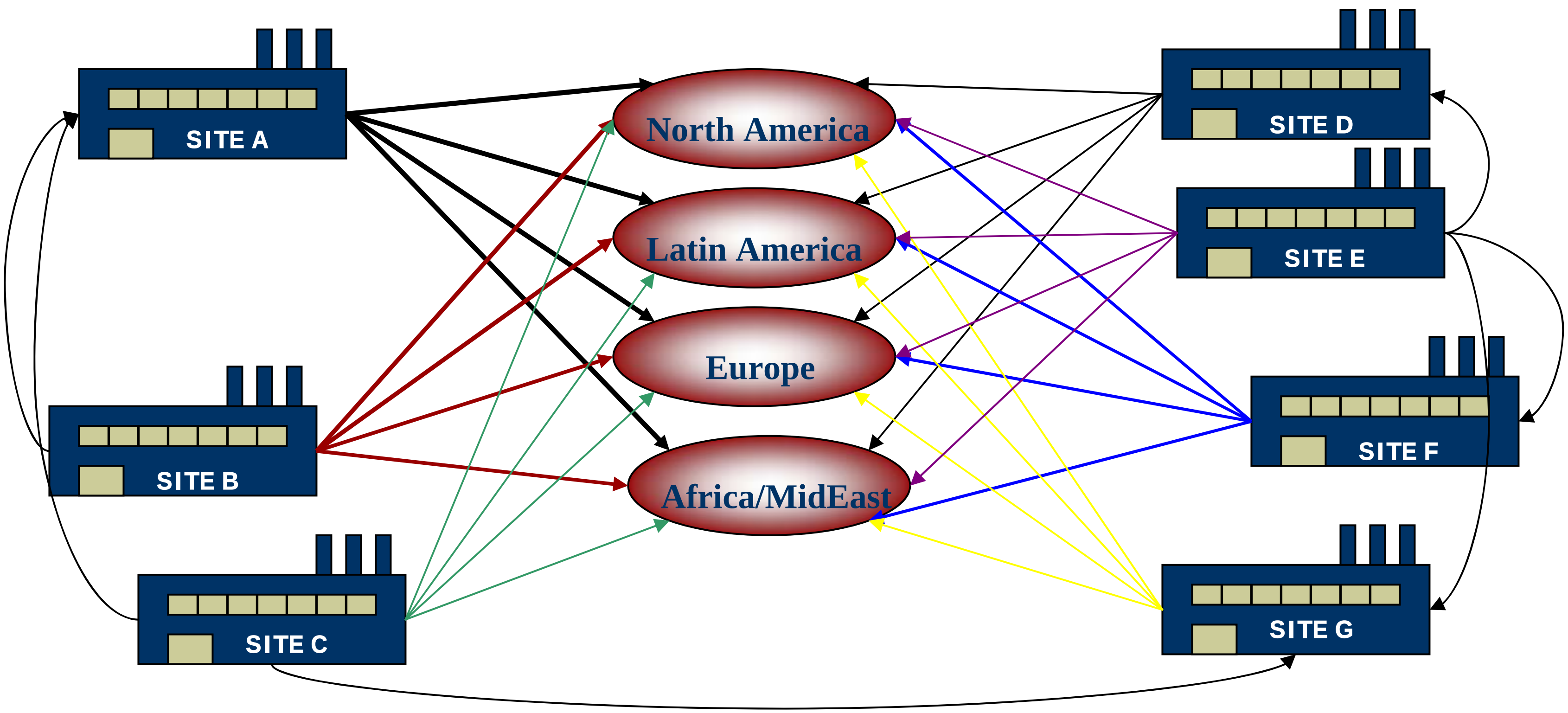
2. Perform branch and bound search where LP relaxation is replaced by Lagrangean relaxation/decomposition to
   a) Obtain tighter bound
   b) Decompose MILP

Typically in Stochastic Programming
Caroe and Schultz (1999)
Goel and Grossmann (2006)
Tarhan and Grossmann (2008)

Remarks
1. Methods can be extended to NLP, MINLP
2. Size of dual gap depends greatly on how problems are decomposed
3. From experience gap often decreases with problem size.

Multisite Distribution Network

Jackson, Grossmann, Wassick, Hoffman (2002)



- **Objective: Develop model and effective solution strategy for large-scale multiperiod planning with *Nonlinear Process Models***

Multisite Distribution Model

•Develop Multisite Model to determine:

1)What products to manufacture in each site

2)What sites will supply the products for each market

3)Production and inventory plan for each site

➢***Objective: Maximize Net Present Value***

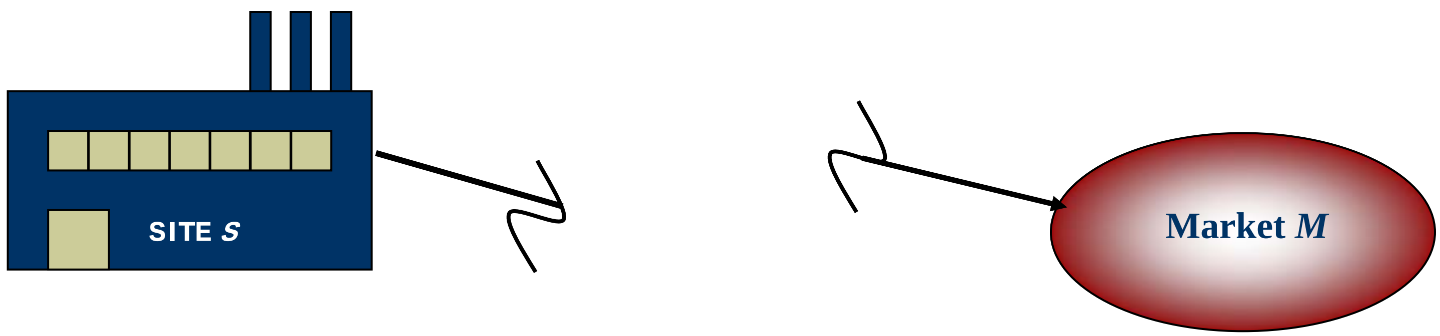
•**Challenges/Optimization Bottlenecks: Large-Scale NLP**

–Interconnections between time periods & sites/markets

➢*Apply **Lagrangean Decompostion** Method*

Spatial Decomposition

$$\max\ PROFIT = SCost_S^{PR,M} * SALES_S^{PR,M} - PCost_S^{PR,M} * PROD_S^{PR,M}$$
$$\boxed{+\ \lambda_S^{PR,M}\left(PROD_S^{PR,M} - SALES_S^{PR,M}\right)}$$



SITE S CONSTRAINTS :

$$f\left(PROD_S^{PR,M}\right) \le 0$$

$$\max\left(-\ PCost_S^{PR,M}\ PROD_S^{PR,M}\ \boxed{+\ \lambda_S^{PR,M}\ PROD_S^{PR,M}}\right)$$

Market M CONSTRAINTS :

$$f\left(SALES_S^{PR,M}\right) \le 0$$

$$\max\left(SCost_S^{PR,M}\ SALES_S^{PR,M}\ \boxed{-\ \lambda_S^{PR,M}\ SALES_S^{PR,M}}\right)$$

Site SUBPROBLEM for all *S* *(NLP)*

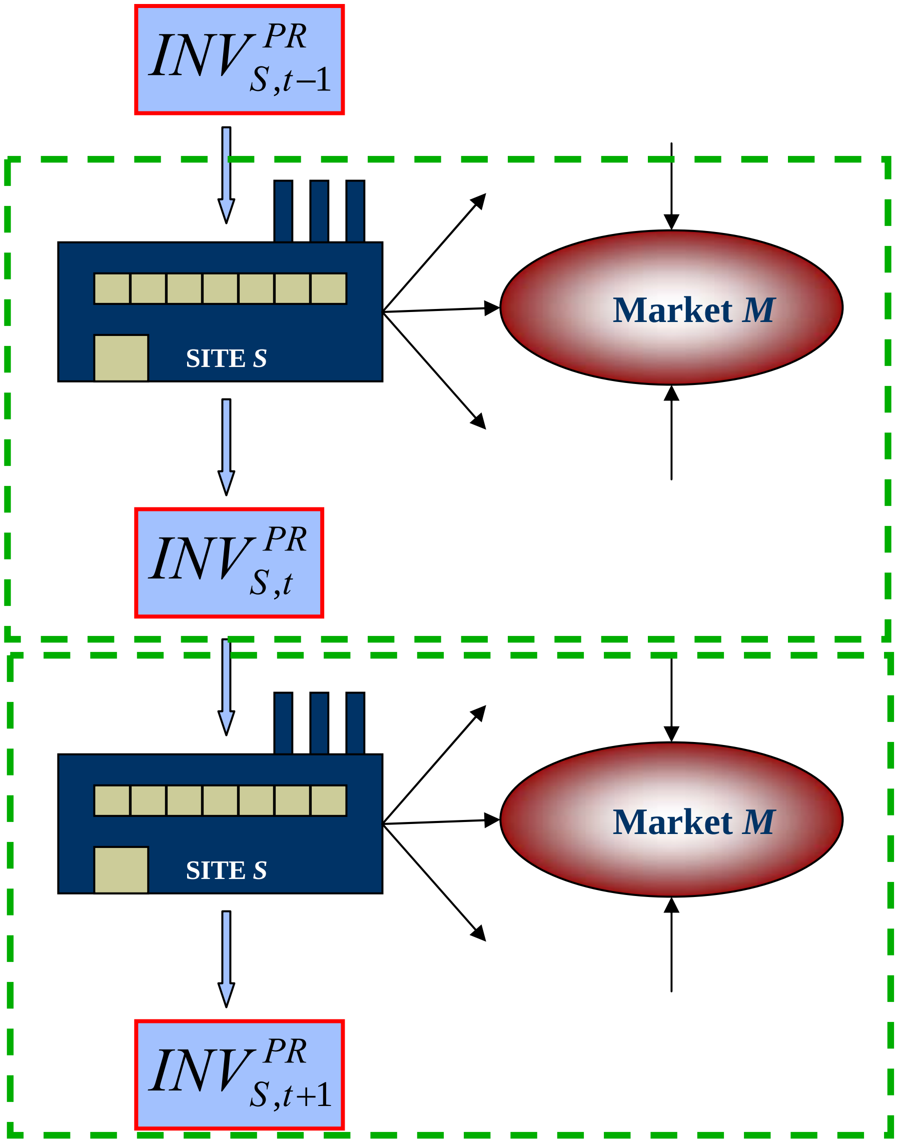
Market SUBPROBLEM for all *M* *(LP)*

Temporal Decomposition

- Decompose at each time period

- Duplicate variables for Inventories for each time period

- Apply Langrangean Decomposition Algorithm



Multisite Distribution Model - Spatial

- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

# Time Periods (months)	Variables/ Constraints	Optimal Solution Profit (million-$)	Full Space Solution Time (CPU sec)	Lagrangean Solution Time (CPU sec)	% Within Full Optimal Solution
2	3345 / 2848	164	52	10	10%
4	6689 / 5698	326	478	127	11%
6	10033 /8548	497	1605	279	9%
8	13377/11398	666	2350	550	9%

Multisite Distribution Model - Temporal

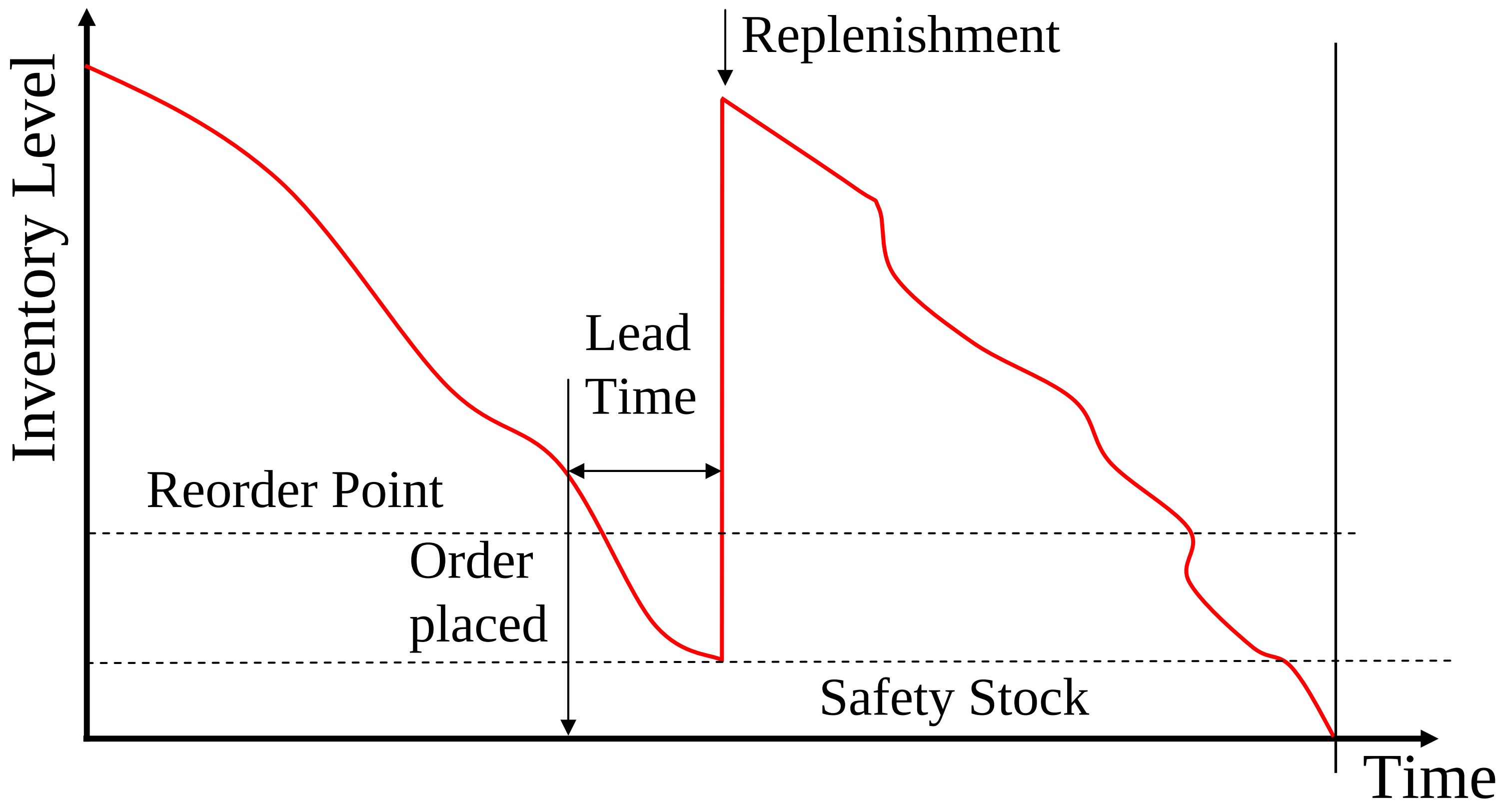
- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

# Time Periods (months)	Variables/ Constraints	Optimal Solution Profit (million-$)	Full Space Solution Time (CPU sec)	Lagrangean Solution Time (CPU sec)	% Within Full Optimal Solution
3	5230 / 5005	116.05	395	97	2.2
6	9973 / 8551	236.53	2013	138	2.3
12	19945 /17101	474.18	10254	278	2.2

Temporal much smaller gap!

Reason: material balances not violated at each time period

Stochastic Inventory System



- Inventory System under Demand Uncertainty
  - Total Inventory = Working Inventory (WI) + Safety Stock (SS)
  - Estimate WI with Economic Order Quantity (EOQ) model

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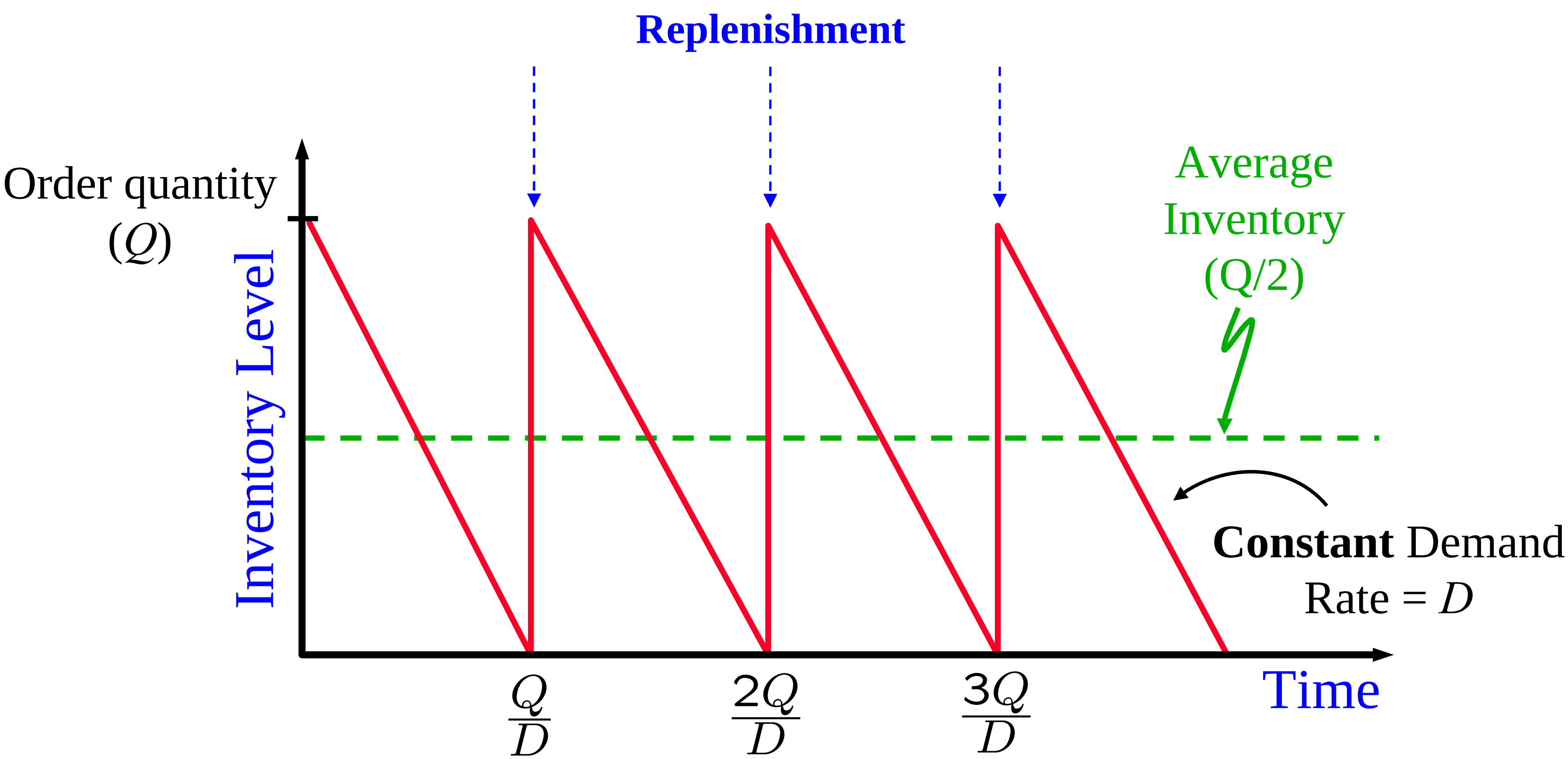
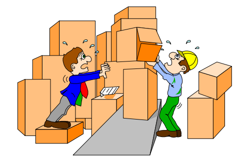
Economic Order Quantity Model



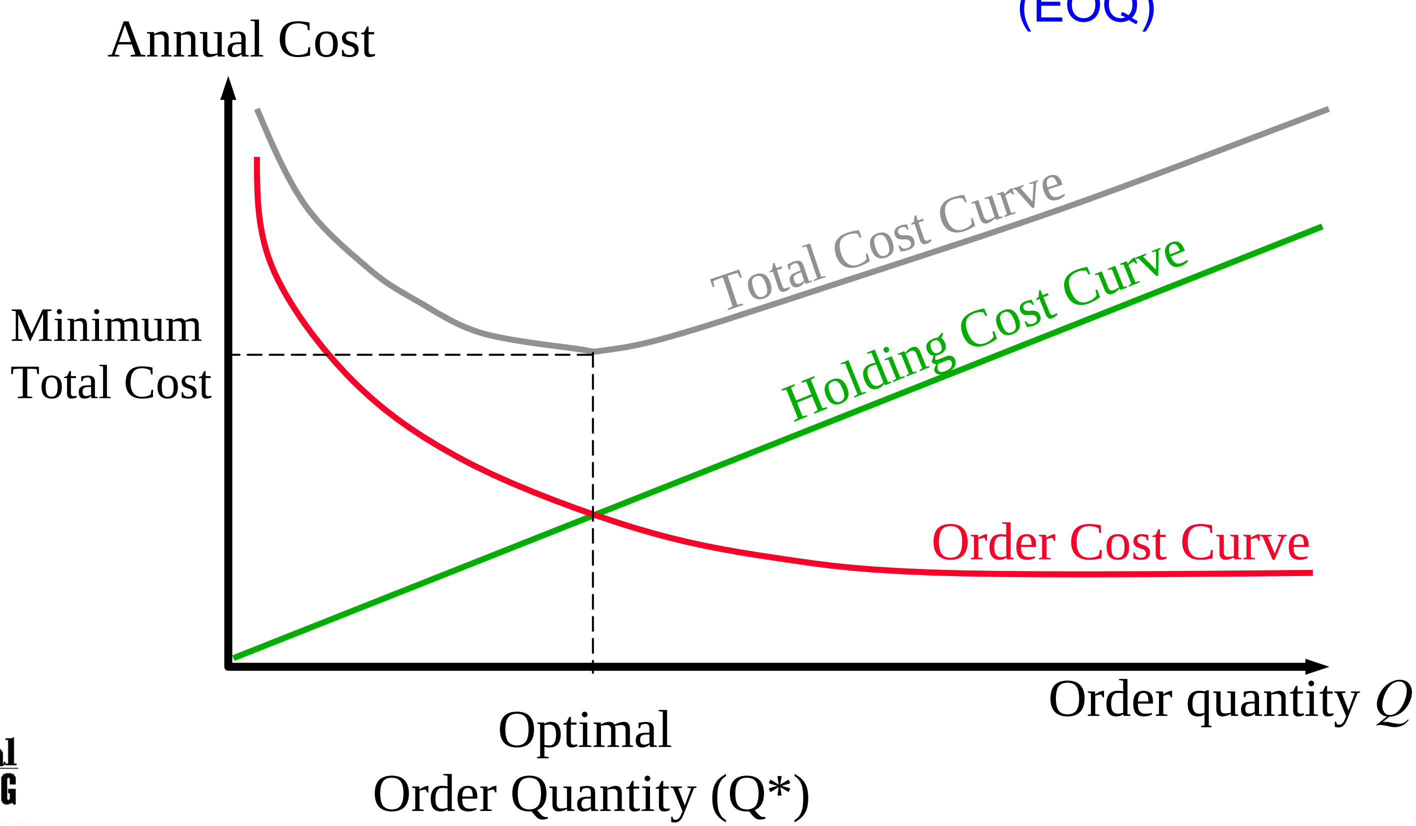
- F = Fixed ordering cost for each replenishment
- h = Unit inventory holding cost

Economic Order Quantity Model

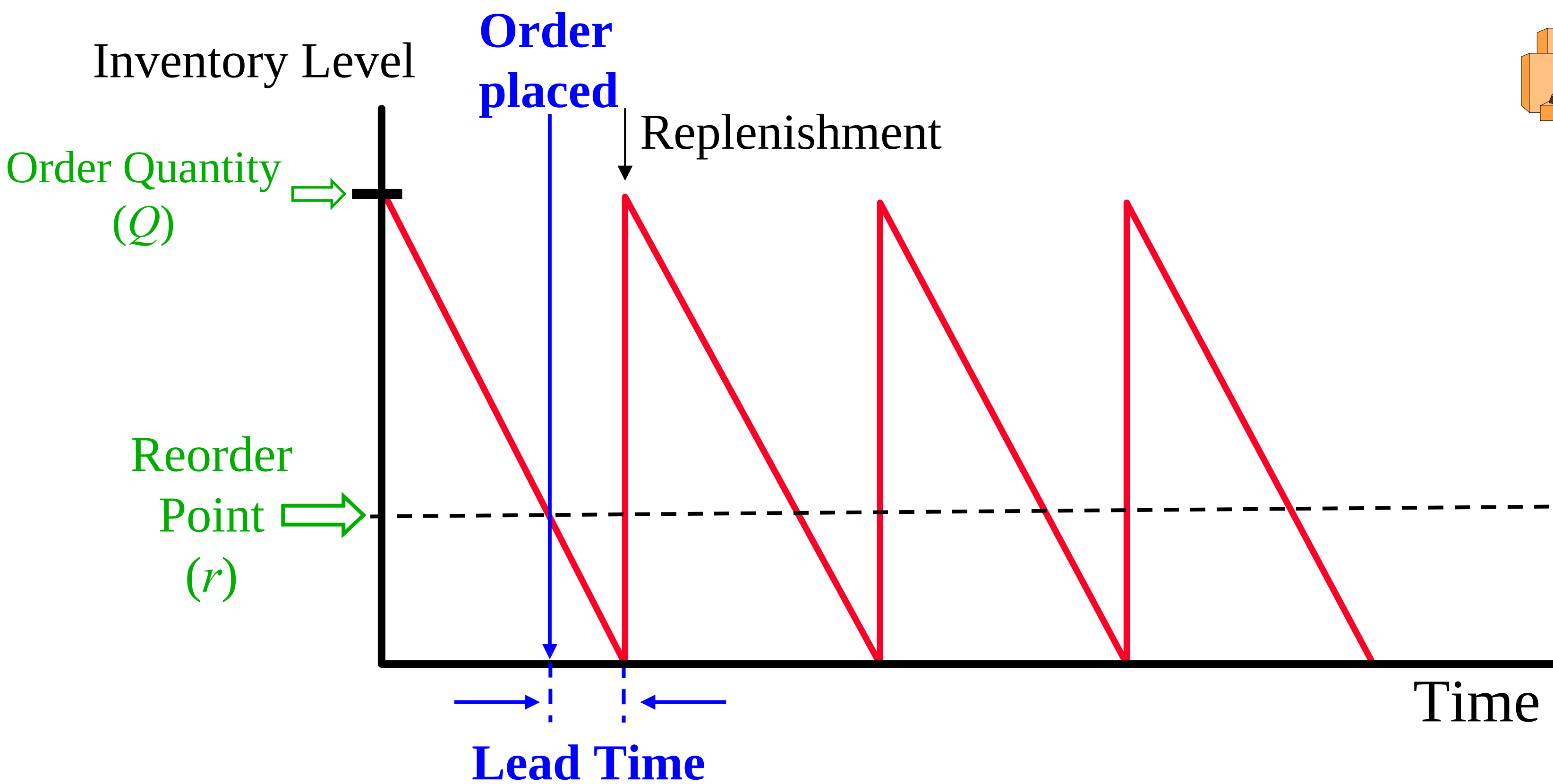


$$\text{Total Cost} = \underbrace{F \cdot \frac{D}{Q}}_{\textit{Order cost}} + \underbrace{h \cdot \frac{Q}{2}}_{\textit{Holding cost}} \Rightarrow \boxed{Q^* = \sqrt{\frac{2FD}{h}}}$$

Economic Order Quantity (EOQ)



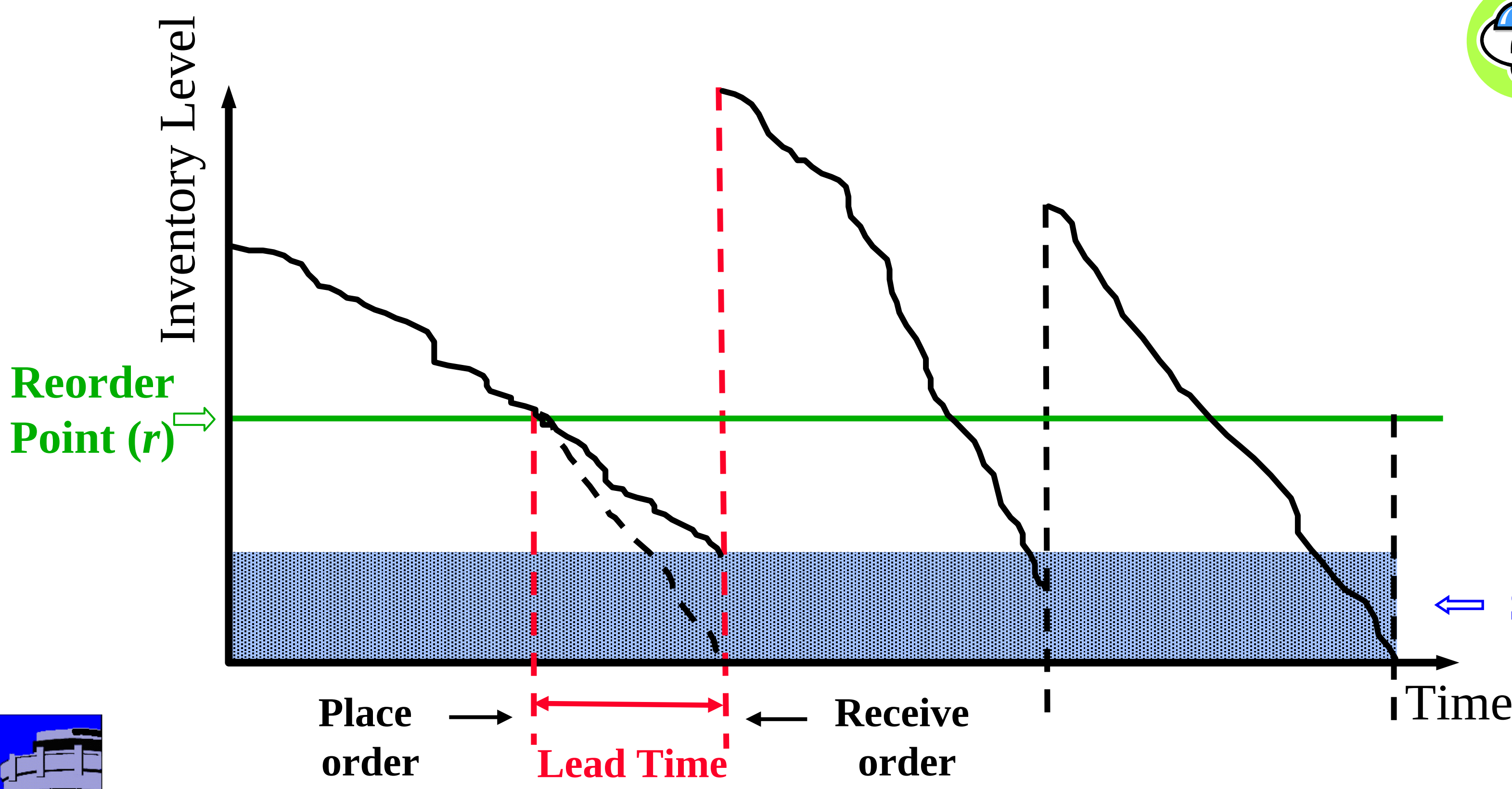
(Q,r) Inventory Policy



- When inventory level falls to r, order a quantity of Q
- Reorder Point (r) = Demand over Lead Time

Stochastic Inventory Model

Stochastic Inventory = Working Inventory (EOQ) + Safety Stock

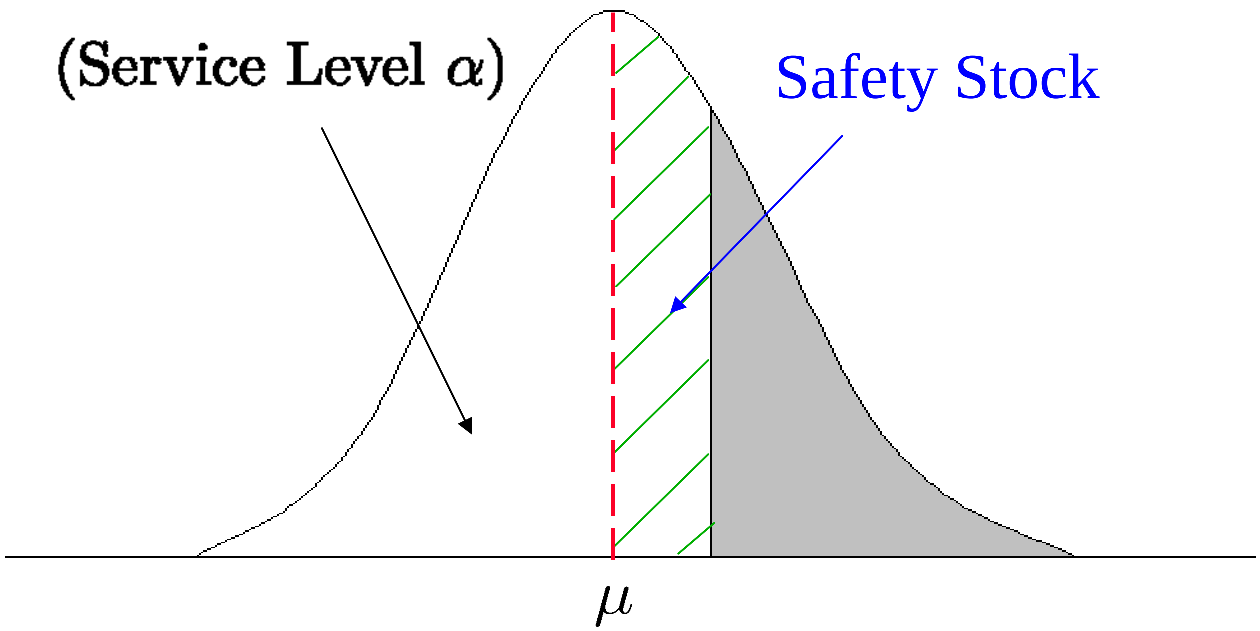


Reorder Point = Expected Demand over Lead Time + Safety Stock

Safety Stock Level

$D \sim N(\mu, \sigma^2) \Rightarrow$ Safety Stock $= \boxed{z_\alpha \sigma}$, $P(z \leq z_\alpha) = \alpha$ **(Service Level)**

Lead time = *L* $\Rightarrow D \sim N(L \cdot \mu, L \cdot \sigma^2) \Rightarrow \boxed{\text{Safety Stock} = z_\alpha \sigma \sqrt{L}}$



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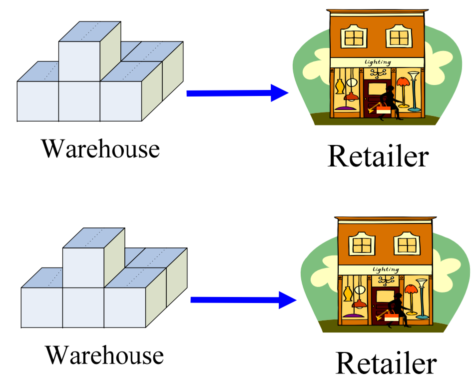
Risk-Pooling Effect*

- Single retailer: $\text{safety stock} = z_\alpha \sigma$

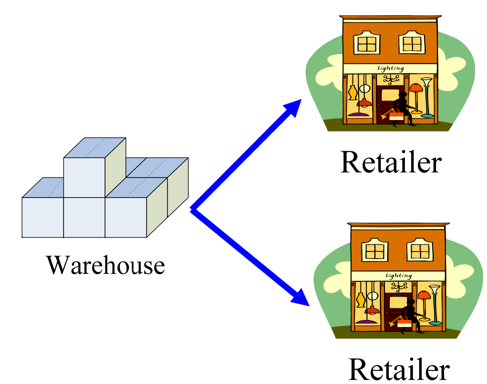
- Decentralized system:
  - Each retailer maintains its own inventory
  - Demand at each retailer is $D_i \sim N(\mu_i, \sigma_i^2)$

$$\boxed{\text{safety stock} = z_\alpha \sum_{i=1}^{N} \sigma_i}$$



- Centralized system:
  - All retailers share common inventory
  - Integrated demand $\sum_i D_i \sim N(\sum_i \mu_i, \sum_i \sigma_i^2)$

$$\boxed{\text{safety stock} = z_\alpha \sqrt{\sum_{i=1}^{N} \sigma_i^2}}$$

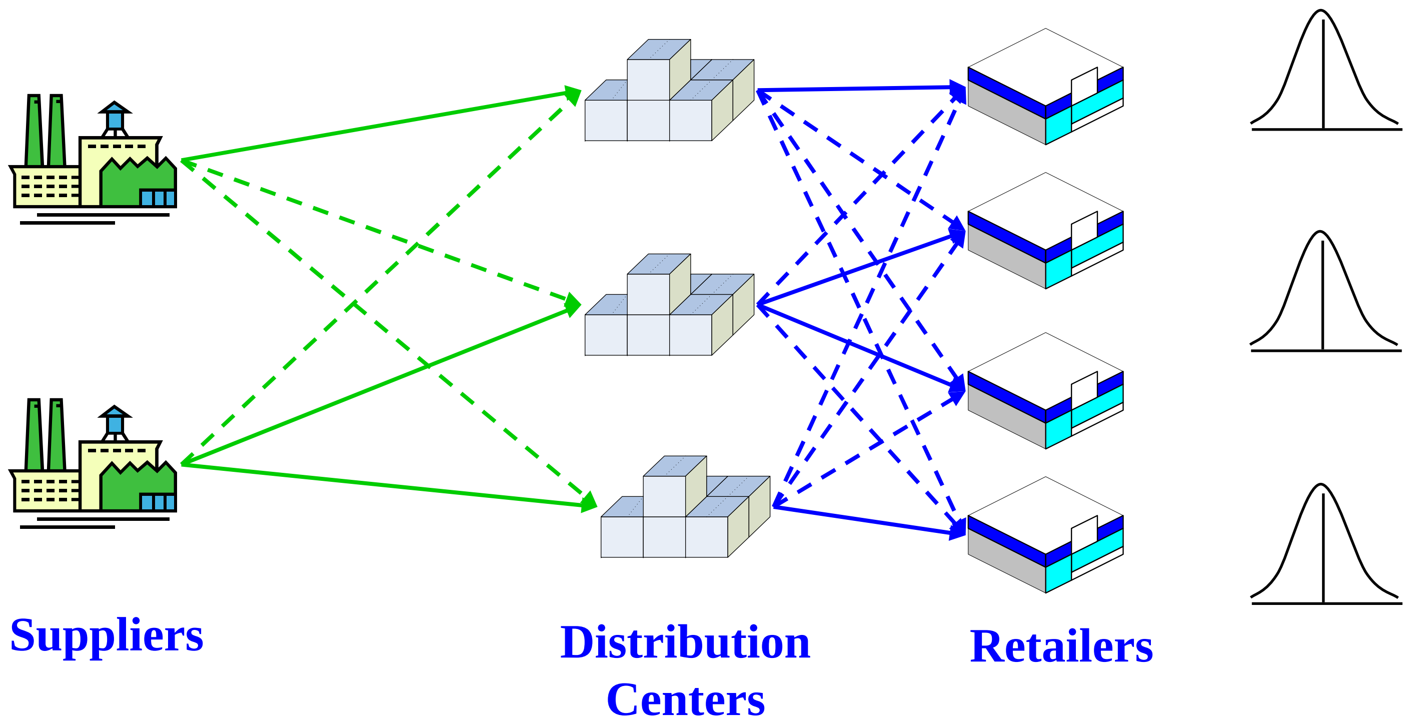


** GD Eppen, "Effect of centralization in a multi-location newsboy problem", Management Science, 1979, 25(5), 498*

Supply Chain Design with Stochastic Inventory Management

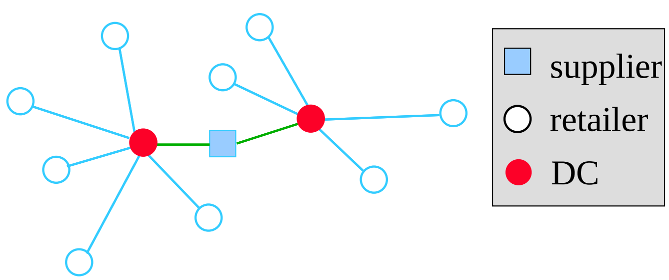
You, Grossmann (2008)

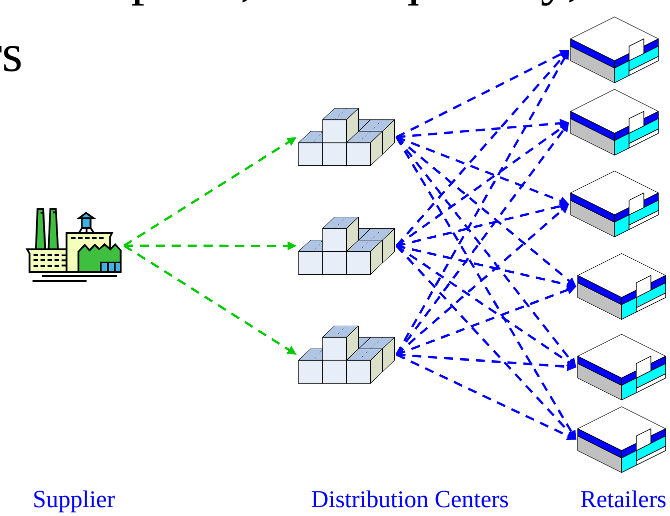
- Given: A potential supply chain
  - Including fixed suppliers, retailers and potential DC locations
  - Each retailer has uncertain demand, using (Q, r) policy
  - Assume all DCs have identical lead time L (lumped to one supplier)



Problem Statement

- Objective: (Minimize Cost)
  - Total cost = DC installation cost + transportation cost + fixed order cost + working inventory cost + safety stock cost
- Major Decisions (Network + Inventory)
  - Network: number of DCs and their locations, assignments between retailers and DCs (single sourcing), shipping amounts
  - Inventory: number of replenishment, reorder point, order quantity, neglect inventories in retailers





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EOQ cost

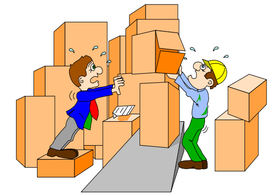
$$\begin{aligned}
D &= \text{expected annual demand} = \sum \chi \mu_i Y_{ij} \\
v(x) &= \text{cost for shipping a order of size } x \text{ from supplier} \\
h &= \text{unit inventory holding cost} \\
F &= \text{fixed cost for place an order} \\
n &= \text{number of orders per year} \\
\chi &= \text{days per year} \\
L &= \text{order lead time} \\
\beta &= \text{weighted factor associated with the transportation cost} \\
\theta &= \text{weighted factor associated with the inventory cost}
\end{aligned}$$

Annual EOQ cost at a DC:

$v(x) = g + ax$

$$\boxed{\underbrace{F \cdot n}_{\text{ordering cost}} + \underbrace{\beta \cdot n \cdot v(\tfrac{D}{n})}_{\text{transportation cost}} + \underbrace{\theta \cdot h \cdot \frac{D}{2n}}_{\text{Working inventory cost}}}$$

Working Inventory Cost



Annual working inventory cost at a DC:

$$\underbrace{F \cdot n + \beta \cdot n \cdot \left(g + \frac{aD}{n}\right) + \theta \frac{h \cdot D}{2n}}_{\text{Convex Function of } n}$$

ordering cost transportation cost inventory cost

Convex Function of *n*

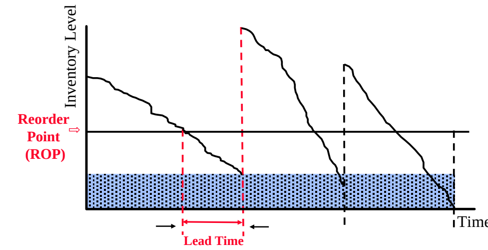
The optimal number of orders is:

$$n^* = \sqrt{((\theta h D)/2(F + \beta g))}$$

The optimal annual EOQ cost:

$$Fn + \beta v\left(\frac{D}{n}\right)n + \theta \frac{hD}{2n} = \boxed{\sqrt{2\theta h D (F + \beta g)} + \beta a D}$$

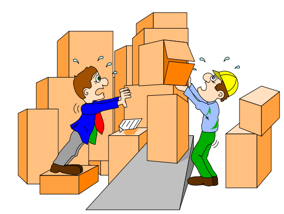
Safety Stock Cost for DCs



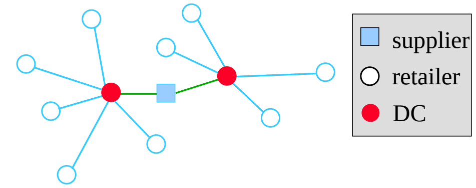
- Demand at retailer $i \sim N(\mu_i, \sigma^2_i)$
- Centralized system (risk-pooling)
- Expected annual cost of safety stock at a DC is:

$$\text{safety stock cost} = \boxed{h \cdot z_\alpha \sqrt{L \sum_{i \in I} \sigma_i^2}}$$

where z_a is the standard normal deviate for which $P(z \leq z_\alpha) = \alpha$



Other Parameters and Variables



I	set of retailers (DC) indexed by i
f_j	fixed (annual)cost of locating a DC at retailer $j, j \in I$
d_{ij}	cost per unit to ship from DC j to retailer i

$$X_j = \begin{cases} 1 & \text{if retailer } j \text{ is selected as a DC} \\ 0 & \text{if not} \end{cases}$$

$$Y_{i,j} = \begin{cases} 1 & \text{if retailer } i \text{ is served by DC based on retailer } j \\ 0 & \text{if not} \end{cases}$$

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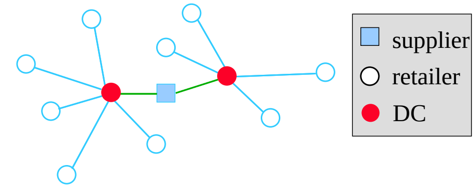
INLP Model Formulation



$$\min \quad \sum_{j \in J} f_j X_j \qquad \text{DC installation cost}$$

$$+ \; \beta \sum_{j \in J} \sum_{i \in I} d_{ij} \chi \mu_i Y_{ij} \qquad \text{DC – retailer transportation}$$

$$+ \; \sum_{j \in J} \sqrt{2\theta h (F_j + \beta g_j) \sum_{i \in I} \chi \mu_i Y_{ij}} + \beta \sum_{i \in I} \sum_{j \in J} (a_j \chi \mu_i Y_{ij}) \qquad \text{EOQ}$$

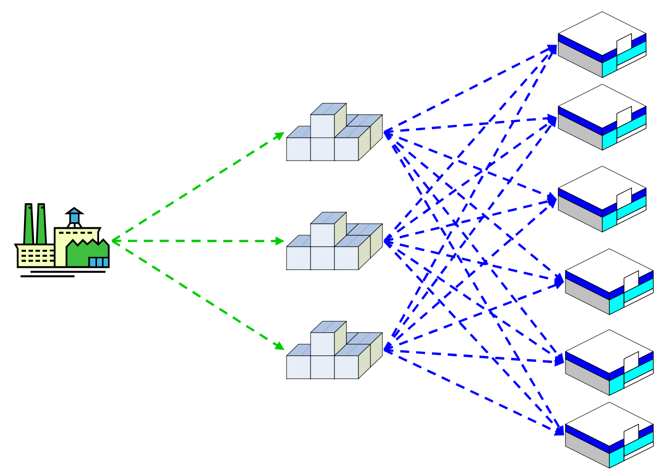
$$+ \; \sum_{j \in J} \Big(\theta h z_\alpha \sqrt{\sum_{i \in I} \sigma_i^2 L_i Y_{ij}}\Big) \qquad \text{Safety Stock}$$

$$\text{s.t.} \quad \sum_{j \in J} Y_{ij} = 1 \quad , \quad \forall i \in I$$

$$Y_{ij} \le X_j \quad , \quad \forall i \in I, j \in J$$

$$X_j, Y_{ij} \in \{0, 1\} \quad , \quad \forall i \in I, j \in J$$

Assignments



Nonconvex INLP

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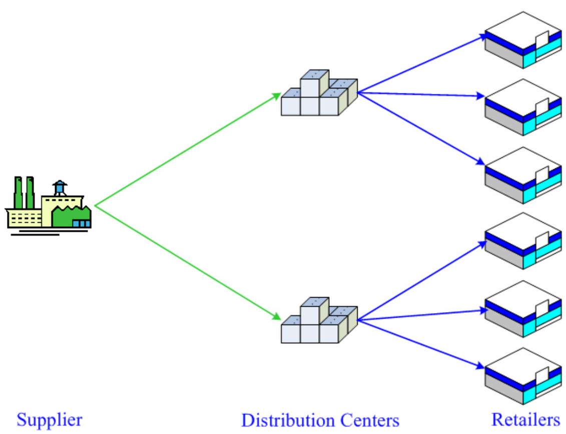
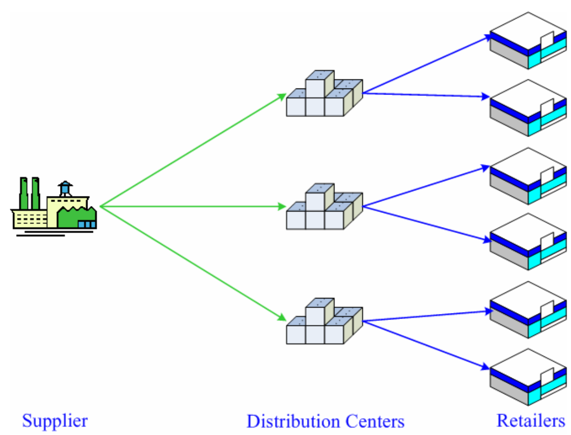
Illustrative Example

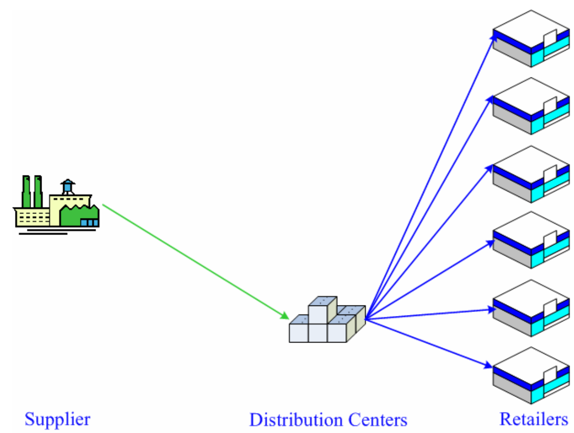
- Small Scale Example
  - A supply chain includes 3 potential DCs and 6 retailers (pervious slide)
  - Different weights for transportation (β) and inventory (θ)



$\beta = 0.01, \theta = 0.01$



$\beta = 0.1, \theta = 0.01$



$\beta = 0.01, \theta = 0.1$

- Model Size for Large Scale Problem
  - INLP model for 150 potential DCs and 150 retailers has 22,650 binary variables and 22,650 constraints – need effective algorithm to solve it …

Model Properties

- Variables Y_{ij} can be relaxed as continuous variables (MINLP)
  - Local or global optimal solution always have all Y_{ij} at integer
  - If $h=0$, it reduces to an "uncapacitated facility location" problem
  - NLP relaxation is very effective (usually return integer solutions)

$$\min \quad \sum_{j\in J} f_j X_j + \sum_{i\in I}\sum_{j\in J} \hat{d_{ij}} Y_{ij} + \sum_{j\in J} K_j \underbrace{\sqrt{\sum_{i\in I} \mu_i Y_{ij}}}_{Z1_j} + \sum_{j\in J} q \underbrace{\sqrt{\sum_{i\in I} \hat{\sigma_i}^2 Y_{ij}}}_{Z2_j}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j\in J} Y_{ij} = 1 \quad , \quad \forall i \in I \\
& Y_{ij} \le X_j \quad , \quad \forall i \in I, j \in J \\
& Y_{ij} \ge 0 \quad , \quad \forall i \in I, j \in J \\
& X_j, \in \{0,1\} \quad , \quad \forall j \in J
\end{aligned}$$

Avoid unbounded gradient

Non-convex MINLP

$$\text{where } \hat{d_{ij}} = \beta\mu_i(d_{ij}+a_j),\ \hat{\sigma_i}^2 = L\sigma_i^2\ K_j = \sqrt{2\theta h(F_j+\beta g_j)},\ q = \theta h z_\alpha$$

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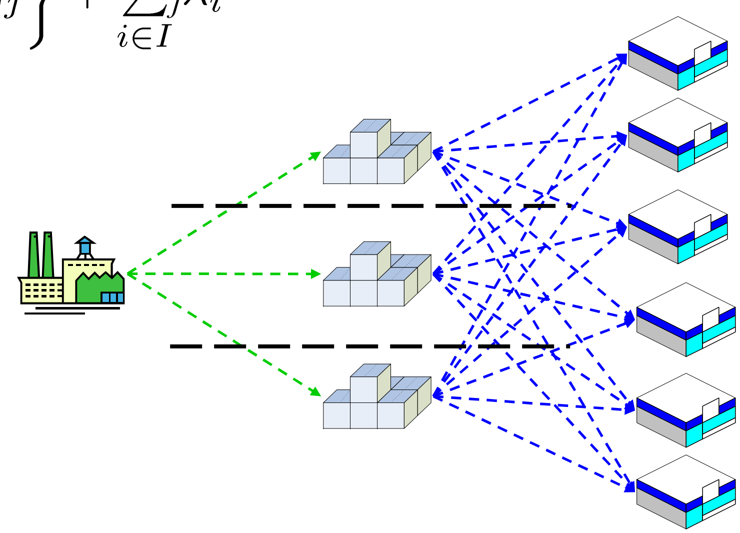
Lagrangean Relaxation

- Lagrangean Relaxation (LR) and Decomposition
  - LR: dualizing the single sourcing constraint: $\sum_{j\in J} Y_{ij} = 1, \forall i \in I$
  - Spatial Decomposition: decompose the problem for each potential DC j
  - Implicit constraint: at least one DC should be installed, $\sum_{j\in J} X_j \geq 1$
    - Use a special case of LR subproblem that $\sum_j X_j = 1$

$$\min \sum_{j\in J} \left\{ f_j X_j + \sum_{i\in I} (\hat{d}_{ij} - \lambda_i) Y_{ij} + K_j Z1_j + q\, Z2_j \right\} + \sum_{i\in I} \lambda_i$$

$$
\begin{array}{rrcl}
\text{s.t.} & Y_{ij} \leq X_j & , & \forall j \in J, \; i \in I \\
& Y_{ij} \geq 0 & , & \forall j \in J, \; i \in I \\
& X_j \in \{0,1\} & , & \forall j \in J \\
& -Z1_j^2 + \sum_{i\in I} \mu_i Y_{ij} \leq 0 & , & \forall j \in J \\
& -Z2_j^2 + \sum_{i\in I} \hat{\sigma}_i^{\,2} Y_{ij} \leq 0 & , & \forall j \in J \\
& Z1_j \geq 0, \; Z2_j \geq 0 & , & \forall j \in J
\end{array}
$$

decompose by DC j



Computational Results

- 88 ~150 retailers
  - Each instance has the same number of potential DCs as the retailers

No. Retailers	β	θ		**Lagrangean Relaxation** (Algorithm 2)						**BARON** (global optimum)		
				Upper Bound	Lower Bound	Gap	Iter.	Time (s)		Upper Bound	Lower Bound	Gap
88	0.001	0.1		867.55	867.54	**0.001 %**	21	**356.1**		867.55*	837.68	3.566 %
88	0.001	0.5		1230.99	1223.46	**0.615 %**	24	**322.54**		1295.02*	1165.15	11.146 %
88	0.005	0.1		2284.06	2280.74	**0.146 %**	55	**840.28**		2297.80*	2075.51	10.710 %
88	0.005	0.5		2918.3	2903.38	**0.514 %**	51	**934.85**		3022.67*	2417.06	25.056 %
150	0.001	0.5		1847.93	1847.25	**0.037 %**	13	**659.1**		1847.93*	1674.08	10.385 %
150	0.005	0.1		3689.71	3648.4	**1.132 %**	53	**3061.2**		3689.71*	3290.18	12.143 %

* Suboptimal solution obtained with BARON for 10 hour limit.

Conclusions

1. **Enterprise-wide Optimization area of great industrial interest**
   Great economic impact for effectively managing complex supply chains

2. **Two key components: Planning and Scheduling**
   Modeling challenge:
   Multi-scale modeling (temporal and spatial integration)

3. **Computational challenges lie in:**
   a) Large-scale optimization models (decomposition, grid computing)
   b) Handling uncertainty (stochastic programming)