State Space-based MPC using a Kalman Filter

B. Wayne Bequette

- Quick Review
- State Space Models
  - Unconstrained and Constrained Solutions
- Kalman Filter for Disturbance Estimation
- Summary
Model Predictive Control (MPC)

- Constraints
- Multivariable
- Time-delays

- Quadratic objective function
- Quadratic program (QP)

- State space model
- Disturbances

additive output vs. state estimation

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Original DMC Approach to Plant-Model Mismatch

\[
\begin{align*}
\hat{x}_{k|k-1} &= \Phi \hat{x}_{k-1|k-1} + \Gamma u_{k-1} \\
\hat{y}_{k|k-1} &= C \hat{x}_{k|k-1}
\end{align*}
\]

Prediction at step k, based on information at k-1

\[
\begin{align*}
\hat{p}_{k|k} &= y_k - \hat{y}_{k|k-1} \\
\hat{y}_{k|k} &= \hat{y}_{k|k-1} + \hat{p}_{k|k}
\end{align*}
\]

Measured output

Model output predicted from k-1

“additive output” disturbance assumption

“additive output” disturbance assumption

Forces the model “corrected output” equal to measured output

Notice that Model States are Not “Corrected”

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1}
\]

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Model Prediction to $k+1$

\[
\hat{x}_{k+1|k} = \Phi \hat{x}_{k|k} + \Gamma u_k
\]

\[
\hat{p}_{k+1|k} = \hat{p}_{k|k}
\]

\[
\hat{y}_{k+1|k} = C\hat{x}_{k+1|k} + \hat{p}_{k+1|k} =
\]

\[
= C\Phi \hat{x}_{k|k} + C\Gamma u_k + \hat{p}_{k+1|k}
\]

\[
= C\Phi \hat{x}_{k|k} + C\Gamma u_{k-1} + C\Gamma \Delta u_k + \hat{p}_{k+1|k}
\]

Assumes future corrections equal to current correction
\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix} = \begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix}
I \\
I \\
\vdots \\
I
\end{bmatrix} \hat{p}_{k|k} + \begin{bmatrix}
C\Gamma \\
C\Phi\Gamma + C\Gamma \\
\vdots \\
\sum_{i=1}^{P} C\Phi^{i-1}\Gamma
\end{bmatrix} u_{k-1}
\]

"free" or "unforced response" (if no more control moves are made)

\[
\begin{bmatrix}
\begin{bmatrix}
C\Gamma \\
C\Phi\Gamma + C\Gamma \\
\vdots \\
\sum_{i=1}^{P} C\Phi^{i-1}\Gamma
\end{bmatrix} & \begin{bmatrix}
0 \\
C\Gamma \\
\vdots \\
\sum_{i=1}^{P-1} C\Phi^{i-1}\Gamma
\end{bmatrix}
\end{bmatrix} \begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

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Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= 
\begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix} \hat{x}_{k|k} 
+ 
\begin{bmatrix}
I \\
I \\
\vdots \\
I
\end{bmatrix} \hat{p}_{k|k} 
+ 
\begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_P
\end{bmatrix} u_{k-1}
\]

"free" or "unforced response" (if no more control moves are made)

\[
\begin{bmatrix}
S_1 & 0 & \cdots & 0 \\
S_2 & S_1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_P & S_{P-1} & \cdots & S_{P-M+1}
\end{bmatrix}
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

\[\Delta u_f\]

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Same Optimization Problem as Before

$$\min_{\Delta u_f} \quad J = \sum_{i=1}^{P} (r_{k+i|k} - \hat{y}_{k+i|k})^T W_y (r_{k+i|k} - \hat{y}_{k+i|k}) + \sum_{i=0}^{M-1} \Delta u_{k+i}^T W^u \Delta u_{k+i}$$

$$\hat{E} = r - \hat{Y} = r - f - S_f \Delta u_f$$

Where

$$W^y = \begin{bmatrix} W^y & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^y \end{bmatrix}$$

$$W^U = \begin{bmatrix} W^u & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^u \end{bmatrix}$$

future setpoints

and

“unforced” error

so

“hidden slide” provided for additional background

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Optimization Problem…

\[
\hat{E}^T W^Y \hat{E} = (E - S_f \Delta u_f)^T W^Y (E - S_f \Delta u_f) \\
= E^T W^Y E - 2\Delta u_f^T S_f^T W^Y E + \Delta u_f^T S_f^T W^Y S_f \Delta u_f
\]

so

\[
\min_{\Delta u_f} \quad J = \hat{E}^T W^Y \hat{E} + \Delta u_f^T W^U \Delta u_f
\]

can be written

\[
\min_{\Delta u_f} \quad J = E^T W^Y E + \Delta u_f^T \left( S_f^T W^Y S_f + W^U \right) \Delta u_f - 2\Delta u_f^T S_f^T W^Y E
\]

and the unconstrained solution is found from

\[
\frac{\partial J}{\partial \Delta u_f} = 0
\]

“hidden slide” provided for additional background

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Unconstrained Solution

Analytical Solution for Unconstrained System

\[ \Delta u_f = \left( S_f^T W^Y S_f + W^U \right)^{-1} S_f^T W^Y E \]

In practice, do not actually invert a matrix. Solve as set of simultaneous equations (or use \ in MATLAB)

\[ \Delta u_f = \left( S_f^T W^Y S_f + W^U \right) \backslash S_f^T W^Y E \]

“hidden slide” provided for additional background
Although a set of control moves is computed, only the first move $\Delta u_k$ is implemented. The next output at $k+1$ is obtained, then a new optimization problem is solved.
Problems with “Classical MPC” (e.g. DMC)

- Finite Step/Impulse Models Limited
  - Limited to open-loop stable processes (there is no corrective feedback to model states)

- Additive Output Disturbance Assumption
  - Poor performance for input step disturbances
  - No explicit measurement noise trade-off

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Process Control: As Taught vs as Practiced
Francis G. Shinskey

LIMITATIONS OF DYNAMIC MATRIX CONTROL

P. Lundström,1† J. H. Lee2‡, M. Morari4§ and S. Skogestad5

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MPC, Including Disturbance Models

\[
x_{k+1} = \Phi x_k + \Gamma u_k + \Gamma^d d_k
\]

with known current state, easy to propagate estimates

\[
x_{k+1} = \Phi x_k + \Gamma u_k + \Gamma^d d_k
\]

\[
y_{k+1} = C x_{k+1} = C \Phi x_k + C \Gamma u_k + C \Gamma^d d_k
\]

and, using control changes

\[
u_k = u_{k-1} + \Delta u_k
\]

\[
y_{k+1} = C \Phi x_k + C \Gamma u_{k-1} + C \Gamma^d d_k + C \Gamma \Delta u_k
\]

Next, use estimated disturbance

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Based on an Estimated Disturbance

\[ \hat{y}_{k+1|k} = C \Phi \hat{x}_{k|k} + C \Gamma u_{k-1} + C \Gamma^d \hat{d}_{k|k} + C \Gamma \Delta u_k \]

Now, propagate the prediction for P steps into the future
Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= \begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix} \begin{bmatrix}
\hat{x}_{k|k} \\
\vdots \\
\hat{x}_{k|k}
\end{bmatrix} + \begin{bmatrix}
C\Gamma \\
C\Phi \Gamma + C\Gamma \\
\vdots \\
\sum_{i=1}^P C\Phi^{i-1} \Gamma
\end{bmatrix} u_{k-1} + \begin{bmatrix}
C\Gamma^d \\
C\Phi \Gamma^d + C\Gamma^d \\
\vdots \\
\sum_{i=1}^P C\Phi^{i-1} \Gamma^d
\end{bmatrix} \hat{d}_{k|k}
\]

"free" or "unforced response" (if no more control moves are made)

Estimated disturbance (constant in future)

\[
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
= \begin{bmatrix}
C\Gamma & 0 & \cdots & 0 \\
C\Phi \Gamma + C\Gamma & C\Gamma & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^P C\Phi^{i-1} \Gamma & \sum_{i=1}^{P-1} C\Phi^{i-1} \Gamma & \cdots
\end{bmatrix} \begin{bmatrix}
\Delta u_f \\
\Delta u_f \\
\vdots \\
\Delta u_f
\end{bmatrix}
\]

"forced" response

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Output Predictions (using S notation)

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= \begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_P
\end{bmatrix} u_{k-1} + \begin{bmatrix}
S_1^d \\
S_2^d \\
\vdots \\
S_P^d
\end{bmatrix} \hat{d}_{k|k}
\]

"free" or "unforced response" (if no more control moves are made)

Estimated disturbance (constant in future)

\[
\begin{bmatrix}
S_1 & 0 & \ldots & 0 \\
S_2 & S_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_P & S_{P-1} & \ldots & S_{P-M+1}
\end{bmatrix}
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

\[\Delta u_f\]
Same Optimization Problem as Before

\[
\min_{\Delta u_f} \ J = \sum_{i=1}^{P} (r_{k+i|k} - \hat{y}_{k+i|k})^T W^y (r_{k+i|k} - \hat{y}_{k+i|k}) + \sum_{i=0}^{M-1} \Delta u_{k+i}^T W^u \Delta u_{k+i}^T
\]

\[
\hat{E}^T W^Y \hat{E}
\]

\[
\Delta u_f^T W^U \Delta u_f
\]

Where

\[
W^Y = \begin{bmatrix}
W^y & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & W^y
\end{bmatrix}
\]

\[
W^U = \begin{bmatrix}
W^u & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & W^u
\end{bmatrix}
\]

future setpoints

\[
\hat{E} = r - \hat{Y} = r - f - S_f \Delta u_f
\]

so

\[
\hat{E} = E - S_f \Delta u_f
\]

“unforced” error

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Optimization Problem...

\[ \hat{E}^T W^Y \hat{E} = \left( E - S_f \Delta u_f \right)^T W^Y \left( E - S_f \Delta u_f \right) \]
\[ = E^T W^Y E - 2 \Delta u_f^T S_f^T W^Y E + \Delta u_f^T S_f^T W^Y S_f \Delta u_f \]

so

\[ \min_{\Delta u_f} J = \hat{E}^T W^Y \hat{E} + \Delta u_f^T W^U \Delta u_f \]

can be written

\[ \min_{\Delta u_f} J = E^T W^Y E + \Delta u_f^T \left( S_f^T W^Y S_f + W^U \right) \Delta u_f - 2 \Delta u_f^T S_f^T W^Y E \]

and the unconstrained solution is found from

\[ \partial J / \partial \Delta u_f = 0 \]
Unconstrained Solution

Analytical Solution for Unconstrained System

$$\Delta u_f = \left( S_f^T W^Y S_f + W^U \right)^{-1} S_f^T W^Y E$$

In practice, do not actually invert a matrix. Solve as set of simultaneous equations (or use \ in MATLAB)

$$\Delta u_f = \left( S_f^T W^Y S_f + W^U \right) \backslash S_f^T W^Y E$$

“unforced” error

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How do we estimate the disturbance?
Concise Review of Optimal Estimation (Kalman Filtering)

- **Kalman & Bucy, 1960**
  

- Developed for well-modeled systems with state input and measurement noise
- Success in Aerospace and other applications

- Problems with initial chemical process applications
  - Limited understanding of limitations
- Bias due to model uncertainty
- Need an “appended state” formulation

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Stochastic Models

\[ x_{k+1} = \Phi x_k + \Gamma u_k + \Gamma^w w_k \]
\[ y_k = C x_k + v_k \]

Input noise:
\[ \text{cov}(w) = Q \]

Measurement noise:
\[ \text{cov}(v) = R \]

Initial State Uncertainty:
\[ \text{cov}(x_0) = P_0 \]
Optimal State Estimate

\[ J(\hat{y}) = \sum_{i=1}^{k} (y_i - C\hat{x})^T R^{-1} (y_i - C\hat{x}) \]
State Estimation Procedure

\[
\hat{x}_{k|k-1} = \Phi \hat{x}_{k-1|k-1} + \Gamma u_{k-1} \quad \text{Prediction}
\]

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k \left( y_k - C \hat{x}_{k|k-1} \right) \quad \text{Correction}
\]

\[
P_k = \Phi P_{k-1} \Phi^T + \Gamma_w Q \Gamma_w^T - \Phi P_{k-1} C^T \left( C P_{k-1} C^T + R \right)^{-1} C P_{k-1} \Phi^T
\]

Kalman gain

\[
L_k = P_{k} C^T \left( C P_{k} C^T + R \right)^{-1}
\]

Discuss steady-state Kalman gain

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State Covariance Comparison

“Open-loop”

\[ P_k = \Phi P_{k-1}\Phi^T + \Gamma_w Q\Gamma^T \]

With measurement (Kalman Filter formulation)

\[ P_k = \Phi P_{k-1}\Phi^T + \Gamma_w Q\Gamma^T - \Phi P_{k-1}C^T \left( CP_{k-1}C^T + R \right)^{-1} CP_{k-1}\Phi^T \]

Measurement update reduces state estimate covariance

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Kalman Filter w/Augmented States

\[
\begin{bmatrix}
    x_{k+1} \\
    d_{k+1} \\
    x^a_{k+1}
\end{bmatrix} = \begin{bmatrix}
    \Phi & \Gamma^d & 0 \\
    0 & I & 0 \\
    \Phi^a & x_k^a & \Gamma^a
\end{bmatrix} \begin{bmatrix}
    x_k \\
    d_k \\
    x_k^a
\end{bmatrix} + \begin{bmatrix}
    \Gamma \\
    0 \\
    \Gamma^{a,w}
\end{bmatrix} u_k + \begin{bmatrix}
    0 \\
    I \\
    0
\end{bmatrix} w_k
\]

\[
y_k = \begin{bmatrix}
    C \\
    C^a \\
    x_k^a
\end{bmatrix} \begin{bmatrix}
    x_k \\
    d_k \\
    x_k^a
\end{bmatrix} + v_k
\]

Predictor-corrector equations:

\[
\hat{x}^a_{k|k-1} = \Phi^a \hat{x}^a_{k-1|k-1} + \Gamma^a u_{k-1}
\]

\[
\hat{x}^a_{k|k} = \hat{x}^a_{k|k-1} + L_k \left( y_k - C^a \hat{x}^a_{k|k-1} \right)
\]

Process noise \( \text{cov}(w) = Q \)

Augmented state (includes disturbance) \( \text{cov}(v) = R \)

Manipulated input

Aug. state estimate

Kalman gain

Measured output

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Disturbance Portion of Augmented State

\[ \hat{x}_{k|k}^a = \begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \end{bmatrix} \]

Constant Disturbance Into the Future

\[ \hat{d}_{k+j|k} = \hat{d}_{k|k} \]

Can use physical knowledge to make better assumptions (periodic effects, etc.)

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Example

\[ g_p(s) = \frac{1}{10s + 1} \]

Comparison of DMC and KF-based MPC for a step input disturbance
Step Input Disturbance

DMC vs. KF-MPC

- $y$ vs. time
- $u$ vs. time
Measurement Noise

DMC vs. KF-MPC

- $y$ vs. time
- $u$ vs. time

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Substantial Measurement Noise
Constrained MPC

Input constraints

\[ u_{\text{min}} \leq u_{k+i} \leq u_{\text{max}} \]

\((n_u x 1)\) \hspace{1cm} \((n_u x 1)\) \hspace{1cm} \((n_u x 1)\)

where

\[ u_k = u_{k-1} + \Delta u_k \]

\[ u_{k+1} = u_{k-1} + \Delta u_k + \Delta u_{k+1} \]

Input “velocity” constraints

\[ \Delta u_{\text{min}} \leq \Delta u_{k+i} \leq \Delta u_{\text{max}} \]

Output constraints

\[ y_{\text{min}} \leq \hat{y}_{k+i|k} \leq y_{\text{max}} \]
MPC: Constraint Formulation

Input

\[
\begin{bmatrix}
    u_{\text{min}} \\
    u_{\text{min}} \\
    \vdots \\
    u_{\text{min}}
\end{bmatrix}
\leq
\begin{bmatrix}
    u_{k-1} \\
    u_{k-1} \\
    \vdots \\
    u_{k-1}
\end{bmatrix}
+ \begin{bmatrix}
    I & 0 & 0 & \cdots & 0 \\
    I & I & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    I & I & I & \cdots & I
\end{bmatrix}
\begin{bmatrix}
    \Delta u_k \\
    \Delta u_{k+1} \\
    \vdots \\
    \Delta u_{k+M-1}
\end{bmatrix}
\leq
\begin{bmatrix}
    u_{\text{max}} \\
    u_{\text{max}} \\
    \vdots \\
    u_{\text{max}}
\end{bmatrix}
\]

Velocity

\[
\begin{bmatrix}
    \Delta u_{\text{min}} \\
    \Delta u_{\text{min}} \\
    \vdots \\
    \Delta u_{\text{min}}
\end{bmatrix}
\leq
\begin{bmatrix}
    \Delta u_k \\
    \Delta u_{k+1} \\
    \vdots \\
    \Delta u_{k+M-1}
\end{bmatrix}
\leq
\begin{bmatrix}
    \Delta u_{\text{max}} \\
    \Delta u_{\text{max}} \\
    \vdots \\
    \Delta u_{\text{max}}
\end{bmatrix}
\]
Output constraints

\[ y_{\text{min}} - f \leq S_f \Delta u_f \leq y_{\text{max}} - f \]

“Free” or “unforced” response

\[ f = \begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} C\Gamma \\ C\Phi \Gamma + C\Gamma \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1} \Gamma \end{bmatrix} u_{k-1} + \begin{bmatrix} C\Gamma^d \\ C\Phi \Gamma^d + C\Gamma^d \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1} \Gamma^d \end{bmatrix} \hat{d}_{k|k} \]

"free" or "unforced response" (if no more control moves are made)
QP Codes: “one-sided” form

\[
\begin{align*}
A_1 &= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\
I & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
I & I & I & \cdots & I \\
\end{bmatrix} \begin{bmatrix} \Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1} \\
\end{bmatrix} \\
&\geq \begin{bmatrix} u_{\text{min}} - u_{k-1} \\
u_{\text{min}} - u_{k-1} \\
\vdots \\
u_{\text{min}} - u_{k-1} \\
\end{bmatrix}
\end{align*}
\]

\[
\Delta u_f 
\]

\[
A_2 &= -\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\
I & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
I & I & I & \cdots & I \\
\end{bmatrix} \begin{bmatrix} \Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1} \\
\end{bmatrix} \\
&\geq \begin{bmatrix} u_{k-1} - u_{\text{max}} \\
u_{k-1} - u_{\text{max}} \\
\vdots \\
u_{k-1} - u_{\text{max}} \\
\end{bmatrix}
\]

\[
\Delta u_f 
\]

\[
b_1 
\]

\[
b_2 
\]

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“one-sided” form for input constraints

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\Delta u_f \geq \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

“one-sided” form for output constraints

\[
S_f \Delta u_f \geq y_{\text{min}} - f
\]
\[
-S_f \Delta u_f \geq -y_{\text{max}} + f
\]

Combining all constraints

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{bmatrix}
\Delta u_f \geq \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]

\[
A \Delta u_f \geq b
\]
Quadratic Program (QP)

Objective Function

\[
\min_{\Delta u_f} \Phi = \frac{1}{2} \cdot \Delta u_f^T H \Delta u_f + c^T \Delta u_f
\]

Decision variables

\[
A \Delta u_f \geq b
\]

\[
\Delta u_{\text{min}} \leq \Delta u_f \leq \Delta u_{\text{max}}
\]

Where

\[
H = S_f^T W^Y S_f + W^U
\]

\[
c = -S_f^T W^Y E
\]
Inverse Response Process: Van de Vusse Reaction

\[ \frac{dCa}{dt} = -k_1Ca - k_3Ca^2 + (Cain - Ca)u \]

\[ \frac{dCb}{dt} = k_1Ca - k_2Cb - Cbu \quad \text{where} \quad u = F/V \]
Output Constraints: Inverse Response Example

Unconstrained

Unconstrained y > -0.05

Less control action in order to meet output constraint
Output Constraints

- May not exist a feasible solution
- Relax constraints until feasible
- “Soft constraints” – Penalize in Objective Function
Unstable CSTR Example

plant output - w/o noise

DMC
Artifact of perfect model

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Effect of Model Error

unstable CSTR - effect of model error

- $y$ vs. time
- $u$ vs. time

- DMC
- KF-MPC

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Multivariable Examples

- Have shown clear advantages of KF-based MPC over DMC for SISO
  - Disturbance rejection
  - Unstable processes
- In general, MPC is especially powerful for handling constrained, multivariable systems (including “nonsquare”)
- Two “simple” example systems follow
  - Two-input, one-output (parallel valves)
  - Quadruple tank (two-input, two-output)
2 input – 1 output

\[ y(s) = \frac{1}{2s + 1} u_1(s) + \frac{2}{2s + 1} u_2(s) + \frac{1}{2s + 1} d(s) \]

\[ W^u = \text{diag}(0.1, 0.1), W^y = 1, P = 10, M = 5 \]

Unconstrained: plant output

QP: plant output

Unconstrained

Constrained

Inputs converge to different values

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2 input – 1 output

Setpoint Change, with constraints

Constrained, $\Delta u_2 \leq 0.1$
2 input – 1 output

Disturbance Rejection: unknown step in input

Tight velocity bounds (0.1) on $u_1$
2 inputs – 1 output

- Many input combinations can achieve same output in steady-state
  - Tuning/constraints determine input values
- “Habituating Control” idea can be used
  - Fast, but “expensive” input for immediate disturbance rejection
  - Slow, but “cheap” input for long-term compensation
Flow splits can radically change dynamic behavior.
Quadruple Tank Problem

Operating Point 1

\[ G_1(s) = \begin{bmatrix} \frac{2.6}{62s + 1} & \frac{1.5}{(23s + 1)(62s + 1)} \\ \frac{1.4}{(30s + 1)(90s + 1)} & \frac{2.8}{90s + 1} \end{bmatrix} \]

Operating Point 2

\[ G_2(s) = \begin{bmatrix} \frac{1.5}{63s + 1} & \frac{2.5}{(39s + 1)(63s + 1)} \\ \frac{2.5}{(56s + 1)(91s + 1)} & \frac{1.6}{91s + 1} \end{bmatrix} \]

No time-delays or right-half-plane zeros.
Should we expect similar closed-loop performance?
Setpoint Responses

Operating Point 1

Operating Point 2

Note difference in time scales

“Wrong way” behavior

B. Wayne Bequette
Quadruple Tank Problem

Operating Point 1 – Minimum Phase

\[ G_1(s) = \begin{bmatrix} \frac{2.6}{62s + 1} & \frac{1.5}{(23s + 1)(62s + 1)} \\ \frac{1.4}{(30s + 1)(90s + 1)} & \frac{2.8}{(90s + 1)} \end{bmatrix} \]

\[ z = -0.060 \text{ and } -0.018 \text{ sec}^{-1} \]

Operating Point 2 – Nonminimum Phase (RHPT zero)

\[ G_2(s) = \begin{bmatrix} \frac{1.5}{63s + 1} & \frac{2.5}{(39s + 1)(63s + 1)} \\ \frac{2.5}{(56s + 1)(91s + 1)} & \frac{1.6}{(91s + 1)} \end{bmatrix} \]

\[ z = -0.057 \text{ and } +0.013 \text{ sec}^{-1} \]
Multivariable Systems

- Can have right-half-plane “transmission zeros” even when no individual transfer function has a RHP zero
- Can have individual RHP zeros yet not have a RHPT zero
  - Fine performance when constraints are not active
  - May fail when one constraint becomes active or a loop is “opened”
- Can exhibit “directional sensitivity” – with some setpoint directions much easier to achieve than others
- Some of these MV properties cause challenges independent of control strategy selected
State Space Form not Limiting

- Step Response or Discrete TF can be written in State Space form

- Virtually all recent theoretical results are based on a state space formulation
Summary

- **State Space MPC**
- **Disturbance Models**
  - State/input or/and output disturbance
  - State/input form handles unstable systems
- **Observer Design**
  - Kalman Filter

- **Can Control Outputs that are not measured**
- **Manipulated Input Blocking**
  - Reduce number of decision variables
Stability Proofs: Linear Models

- **Unconstrained**
  - Infinite horizon, State and Input Weights
    - LQ Control
    - Couple with Kalman Filter if some states are unmeasured
  - State penalty at end of horizon

- **Constrained**
  - Terminal state constraint
  - Stabilizing state feedback at end of prediction horizon