(I) MODELING CHEMICAL SYSTEMS

by

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Contents

Modeling Techniques

- First-principles modeling methods
  - Also called physically-based or theoretical modeling
  - Based on *engineering conservation principles*

$$\Sigma \quad \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) & \textit{(state equation)} \\ \boldsymbol{y}(t) = \boldsymbol{g}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) & \textit{(output equation)} \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & \textit{(initial state)} \end{cases}$$

$\boldsymbol{x} \in \Re^n$	state vector	$\boldsymbol{f} \in \Re^{\mathrm{n}}$	state-derivative map
$\boldsymbol{u} \in \Re^p$	input vector	$\boldsymbol{g} \in \Re^{\mathrm{m}}$	output map
$\boldsymbol{y} \in \Re^m$	output vector	$t \in \Re$	time
$\boldsymbol{x}_0 \in \Re^{\mathrm{n}}$	initial state	$t_0 \in \Re$	initial time

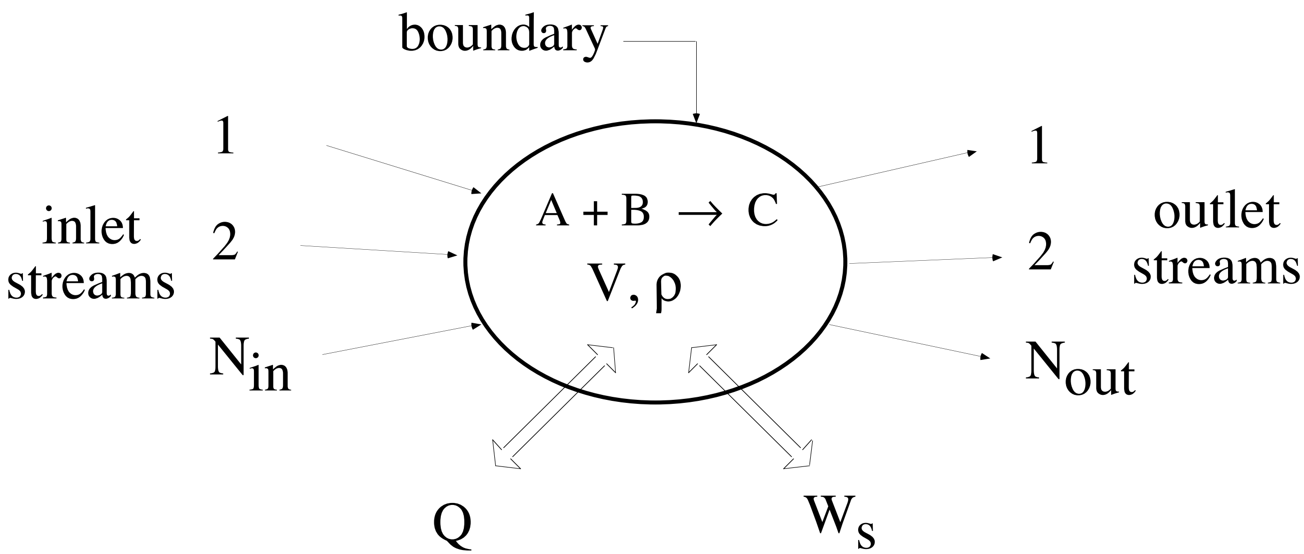
- Black-box modeling

- Response-curve models
  - Also called empirical modeling

- Semi-empirical models
  - Parts of the model are derived from first-principles and other parts are derived from experimental observations or from physical intuition.

Engineering Conservation Principles

- First-principles (or physically based) modeling approach
- Abstraction of an open system:



- Conservation principle

$$\left\{\begin{matrix} \textit{Rate of} \\ \textbf{accumulation} \\ \textit{inside} \\ \textit{volume} \end{matrix}\right\} = \left\{\begin{matrix} \textit{Rate of} \\ \textbf{input} \\ \textit{through} \\ \textit{boundary} \end{matrix}\right\} - \left\{\begin{matrix} \textit{Rate of} \\ \textbf{output} \\ \textit{through} \\ \textit{boundary} \end{matrix}\right\} + \left\{\begin{matrix} \textit{Rate of} \\ \textbf{generation} \\ \textit{inside} \\ \textit{volume} \end{matrix}\right\} - \left\{\begin{matrix} \textit{Rate of} \\ \textbf{depletion} \\ \textit{inside} \\ \textit{volume} \end{matrix}\right\}$$

- Conserved quantities
  - Total mass
  - Species
  - Electronic charge
  - Total energy
  - Momentum
  - Entropy
  - etc.

- **Note**:

  The rate of accumulation is a derivative with respect to time

$$\left\{ \begin{array}{c} \text{rate} \\ \text{of accumulation} \end{array} \right\} = \frac{d}{dt} \left\{ \begin{array}{c} \text{quantity} \\ \text{conserved} \end{array} \right\}$$

  This is the reason why chemical engineering process models are often in the form of a differential equation.

Conservation Equations

Total Mass Balance

$$\frac{d\rho V}{dt} = \sum_{(\text{inlets})\, i=1}^{N_{in}} \rho_i F_i - \sum_{(\text{outlets})\, j=1}^{N_{out}} \rho_j F_j$$

Species Balance

$$\frac{dVC_A}{dt} = \sum_{(\text{inlets})\, i=1}^{N_{in}} C_{A_i} F_i - \sum_{(\text{outlets})\, j=1}^{N_{out}} C_{A_j} F_j - r_A V$$

Energy Balance

$$\frac{d}{dt}\left[\rho V(\hat{U} + \hat{P} + \hat{K})\right] = \sum_{(\text{inlets})\, i=1}^{N_{in}} \rho_i F_i (\hat{H}_i + \hat{P}_i + \hat{K}_i) - \sum_{(\text{outlets})\, j=1}^{N_{out}} \rho_i F_i (\hat{H}_i + \hat{P}_i + \hat{K}_i)$$

$$+ \ Q + (-\Delta\hat{H}_{rxn})\, r_{rxn} V + W_s - P\,\frac{dV}{dt}$$

Common Nomenclature

- **Physical variables**

Volume inside a boundary	V	[l]
Density	ρ	[kg/l]
Volumetric flow rate	F	[l/s]
Mass flow rate	$\boxed{w = \rho F}$	[kg/s]
Linear velocity of flow	v	[m/s]

- **Thermodynamic variables**
  - Temperature — T — [C]
  - Pressure — P — [atm]
  - Specific volume — $\hat{V} = 1/\rho$ — [l/kg]
  - Specific enthalpy — $\hat{H}$ — [J/ kg]
  - Specific internal energy — $\hat{U} = \hat{H} - P\hat{V}$ — [J/ kg]
  - Specific kinetic energy — $\hat{K}$ — [J/ kg]
  - Specific potential energy — $\hat{P}$ — [J/ kg]
  - Specific heat (heat capacity) — C_p — [J/ kg-C]
  - Heat flow rate (Power) — Q — [W] = [J/s]
    - $Q > 0$ heat enters the system
    - $Q < 0$ heat leaves the system
  - Work rate (Power) — W — [W] = [J/s]
    - $W > 0$ work done on the system
    - $W < 0$ work done by the system

- Thermal mass (capacitance) of mass ρV — $\rho V C_p$ — [J/C]

- **Chemical-kinetics variables**
  - Specific heat of reaction — $\Delta \hat{H}_{rxn}$ — [J/mol]
    - $\Delta \hat{H}_{rxn} > 0$ heat consumed by the system (endothermic reaction)
    - $\Delta \hat{H}_{rxn} < 0$ heat produced by the system (exothermic reaction)
  - Overall rate of reaction — r_{rxn} — [mol/sec-l]
  - Rate of reaction of species A — r_A — [mol/sec-l]
  - Concentration of species A — C_A — [mol/l]

Physical Correlations

- Density
  - a. Real Substance
    - $\rho = \rho(T)$ pure liquids
    - $\rho = \rho(T, P)$ pure gases
    - ρ is also a function of composition in mixtures
  - b. Liquids $\boxed{\rho \approx \text{constant}}$ is often a good approximation
  - c. Ideal gases $\boxed{\rho = \dfrac{MP}{RT}}$ where M is the molecular weight

- Specific heat (heat capacity) at constant pressure
  - a. Real Substance Typical correlations of the form:
    - $C_p = C_p(T) = A_1+A_2T+A_3T^{-2}$ [J/kg-C]
    - $C_p = C_p(T) = B_1+B_2T+B_3T^2+B_4T^3$ [J/kg-C]
    - C_p is also a function of composition in mixtures
  - b. Liquids $\boxed{C_p \approx \text{constant}}$ valid for small T variations
  - c. Ideal gases $\boxed{C_p \approx \text{constant}}$

- Specific Enthalpy

  - a. Real Substance (let T^o and P^o be reference states)

$$\hat{H}(T,P) = \hat{H}(T^o,P^o) + \int_{T^o,P^o}^{T,P^o} C_p(T')\,dT' + \int_{T,P^o}^{T,P} \hat{V}\,(1 - T\alpha_T)\,dP'$$

    - $\alpha_T = \frac{1}{\hat{V}}\left(\partial\hat{V}/\partial T\big|_P\right)$ thermal expansion coefficient
    - $\hat{H}(T,P)$ is also a function of composition in mixtures
    - Latent heats must be added as phase transitions are encountered
    - For most liquids, for many real gases, and for all ideal gases, the pressure dependence is negligible, hence:

$$\hat{H}(T) = \hat{H}(T^o) + \int_{T^o,P^o}^{T,P^o} C_\mathrm{p}(T')\,dT'$$

- b. Liquids $\boxed{\hat{H}(T) = \alpha + C_{\mathrm{p,liq}}T \qquad \alpha = constant}$

- c. Ideal gases $\boxed{\hat{H}(T) = \beta + \Delta\hat{\mathrm{H}}_{\mathrm{vap}}(T^{bp}) + C_{\mathrm{p,gas}}T \qquad \beta = constant}$

- Specific Internal Energy

  - a. Real Substance $\hat{\mathrm{U}} = \hat{\mathrm{H}} - \mathrm{P}\hat{\mathrm{V}}$

  - b. Liquids $\boxed{\hat{U}(T) = \alpha + C_{\mathrm{p,liq}}T}$ since for liquids $\mathrm{P}\hat{\mathrm{V}} \ll \hat{\mathrm{H}}$

  - c. Ideal gases $\boxed{\hat{U} = \beta + \Delta\hat{\mathrm{H}}_{\mathrm{vap}} + (C_{\mathrm{p,gas}} - \frac{\mathrm{R}}{\mathrm{MW}})T}$ since $\mathrm{P}\hat{\mathrm{V}} = \frac{\mathrm{RT}}{\mathrm{M}}$

- Specific Kinetic Energy.

  - $$\boxed{\hat{K} = \frac{1}{2} v^2}$$ [m^2/s^2] = [J/kg]

  - v linear velocity of stream (at inlet or outlet) or of the center or of the center of mass of volume V

- Specific Potential Energy

  - $$\boxed{\hat{\mathcal{P}} = g\,(z - z_o)}$$ [m^2/s^2] = [J/kg]

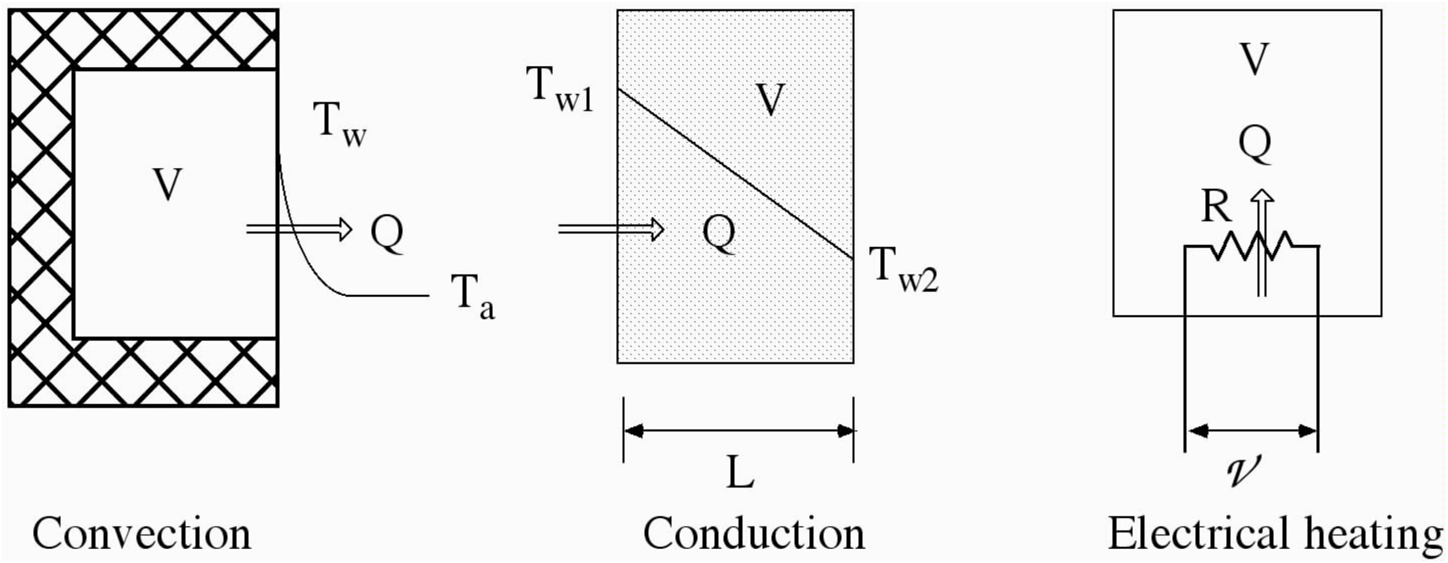
    - g acceleration of gravity
    - z_o height of a reference plane

- Work

  - W_s rate of shaft-work [J/s] = [W]

  - $W_f = -P \dfrac{dV}{dt}$ rate of flow-work [J/s] = [W]

- Heat flow rates



- a. Convection $\boxed{Q = A\, h\, (T_a - T_w)}$ [J/s] = [W]
  - h heat transfer coefficient [W/m2-C]
  - T_w temp. of the boundary wall [C]
  - T_a ambient (bulk) temperature [C]
  - A effective heat-transfer area [m2]

- b. Conduction $\boxed{Q = A k_T \frac{T_{w1} - T_{w2}}{L}}$ [J/s] = [W]
  - k_T thermal conductivity [W/m-C]
  - L slab thickness [m]
- c. Electric heating $\boxed{Q = R I^2 = \frac{\mathcal{V}^2}{R}}$ [J/s] = [W]
  - R resistance [Ω]
  - I current [Amp]
  - $\mathcal{V}$ voltage [V]
- d. Radiation $\boxed{Q \sim T^4}$

- Chemical Kinetics
  - a. Chemical reaction

$$a\ A + b\ B \rightleftharpoons \mathrm{c\ C}$$

  - b. Reaction rate of species A is defined in a closed system:
    - $r_A = \left(\dfrac{dC_A}{dt} \right)_{\substack{\text{closed}\\ \text{system}}}$ [mol/l-s]
      - $r_A < 0$ consumption of A by reaction
      - $r_A > 0$ generation of A by reaction

- c. Phenomenological kinetic models for r_A

  - $$\boxed{r_A = k_r\, C_C^m - k_f\, C_A^s\, C_B^n}$$ [mol/l-s]
  - $k_f = k_{f,o}\, e^{-E_{a,f}/RT}$ k_f: forward kinetic-rate parameter
  - $k_r = k_{r,o}\, e^{-E_{a,r}/RT}$ k_r: reverse kinetic-rate parameter

- d. Overall rate of reaction

  - $$\boxed{r_{rxn} = \left|\frac{r_A}{a}\right| = \left|\frac{r_B}{b}\right| = \left|\frac{r_C}{c}\right| \geq 0}$$ [mol/l-s]

- e. Heat of reaction $\boxed{\Delta H_{rxn} = \Delta H_{rxn}(T)}$ [J/mol]

  - $\Delta H_{rxn} < 0$ exothermic reaction
  - $\Delta H_{rxn} > 0$ endothermic reaction

- f. Examples

  - 1. The irreversible chemical reaction $2A \rightarrow B$

    is known to be of second order. Then

$$r_A = -k_f\, C_A^2 \qquad [\text{ mol/l-s }]$$

$$r_{rxn} = \frac{k_f\, C_A^2}{2} \qquad [\text{ mol/l-s }]$$

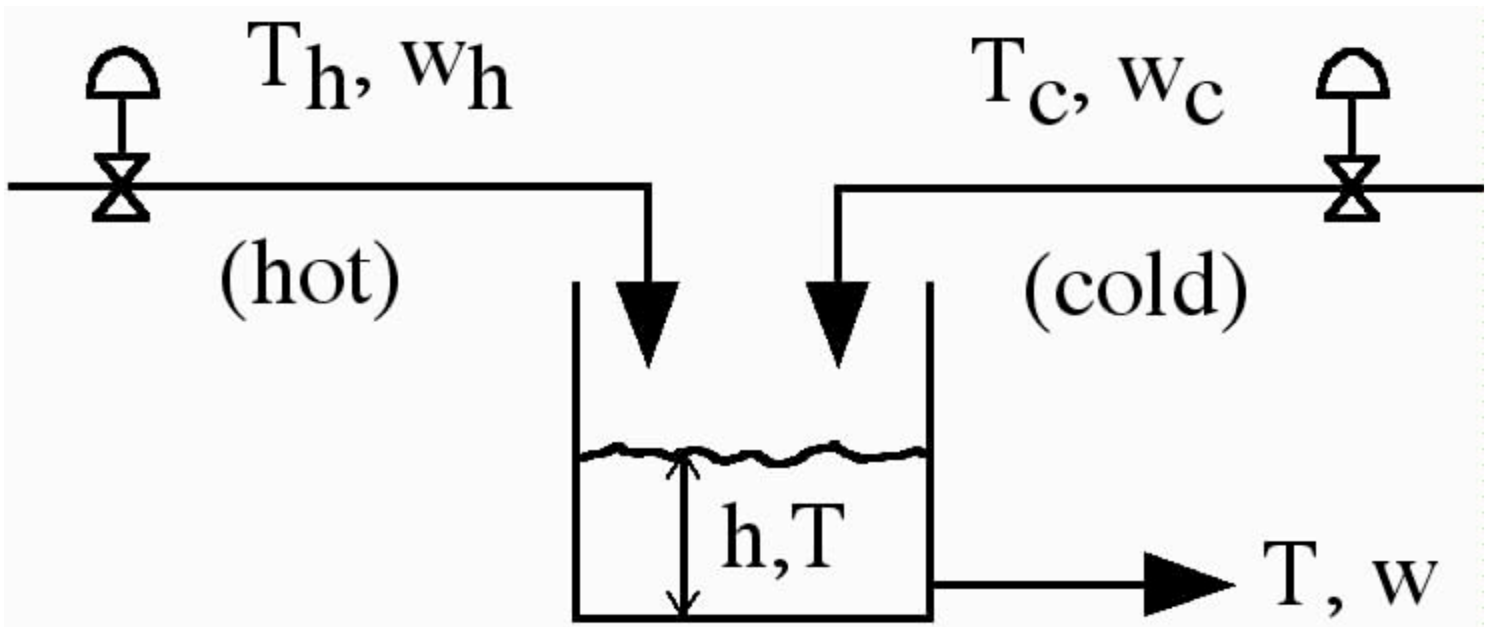
  - 2. The reversible chemical reaction $3\,A + B \rightleftharpoons 2\,C$

    is known to be elementary. Then

$$r_A = k_r\, C_C^2 - k_f\, C_A^2\, C_B \qquad [\text{ mol/l-s }]$$

$$r_{rxn} = \frac{\left| k_r\, C_C^2 - k_f\, C_A^2\, C_B \right|}{3} \qquad [\text{ mol/l-s }]$$

A Modeling Example



T	temperature	A	cross-sectional area
w	mass flow-rate		
h	liquid level	$V = hA$	liquid volume

- **Total Mass Balance**

  - Assumptions

    - Gravity drain $w(t) = \sqrt{h(t)}\big/ R$

  - Balance

$$\frac{d}{dt}[\rho A h(t)] = \rho_h F_h(t) + \rho_c F_c(t) - \rho F(t)$$

$$= w_h(t) + w_c(t) - w(t)$$

$$\frac{dh(t)}{dt} = \frac{1}{\rho A} w_h(t) + \frac{1}{\rho A} w_c(t) - \frac{1}{\rho A} \frac{\sqrt{h(t)}}{R} \qquad (1)$$

- **Energy Balance**
  - Assumptions
    - No heat losses, $C_{p,h} = C_{p,c} = C_p$, $\alpha = 0$
  - Balance

$$\frac{d}{dt}\left(\rho A h(t) C_p T(t)\right) = w_h(t) C_p T_h(t) + w_c(t) C_p T_c(t) - w(t) C_p T(t)$$

$$C_p T(t) \boxed{\frac{d}{dt}\left(\rho A h(t)\right)} + \rho A C_p h(t) \frac{dT(t)}{dt} = w_h(t) C_p T_h(t) + w_c(t) C_p T_c(t) - w(t) C_p T(t)$$

Substitute equation (1) to eliminate $dh(t)/dt$ from the above expression

$$\frac{dT(\mathrm{t})}{dt} = -\frac{1}{\rho A}\frac{w_{\mathrm{h}}(t)T(t)}{h(t)} - \frac{1}{\rho A}\frac{w_{\mathrm{c}}(t)T(t)}{h(t)} + \frac{1}{\rho A}\frac{w_{\mathrm{h}}(t)T_{\mathrm{h}}(t)}{h(t)} + \frac{1}{\rho A}\frac{w_{\mathrm{c}}(t)T_{\mathrm{c}}(t)}{h(t)} \qquad (2)$$

- **Classification of variables**

Variable Type	Physical variable	Std nomenclature
states	h, T	x1, x2
outputs (temperature/level control)	h, T	y1, y2
inputs	w_h, w_c	u1, u2
disturbances	T_h, T_c	d1:=u3, d2:=u4
parameters	ρ, A, C_p, R	p1, p2, p3, p4

- The hot and cold temperatures cannot be manipulated at will, though they may change; hence, they are classified as <u>disturbances</u>
- The hot and cold flow-rates are adjusted as necessary to keep the output temperature at a target value; hence they are classified as <u>inputs</u>.
- Both states are measured, hence the <u>outputs</u> are equal to the states in this case

- **Nonlinear state-space model**
  - State and input vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} h(t) \\ T(t) \end{bmatrix} \in \Re^2 \qquad \boldsymbol{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} w_h(\mathrm{t}) \\ w_c(t) \\ T_h(t) \\ T_c(t) \end{bmatrix} \in \Re^4$$

  - Nonlinear state-space equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) = \begin{bmatrix} f_1\left(\mathrm{x}_1(t), \mathrm{x}_2(t), \mathrm{u}_1(t), \mathrm{u}_2(t), \mathrm{u}_3(t), \mathrm{u}_4(t)\right) \\ f_2\left(\mathrm{x}_1(t), \mathrm{x}_2(t), \mathrm{u}_1(t), \mathrm{u}_2(t), \mathrm{u}_3(t), \mathrm{u}_4(t)\right) \end{bmatrix}$$

where

$$f_1(\boldsymbol{x}, \boldsymbol{u}) = \frac{1}{\rho A} w_h(\mathrm{t}) + \frac{1}{\rho A} w_c(\mathrm{t}) - \frac{1}{\rho A R}\sqrt{h(t)}$$

$$f_2(\boldsymbol{x},\boldsymbol{u}) = -\frac{1}{\rho A}\frac{w_h(t)T(t)}{h(t)} - \frac{1}{\rho A}\frac{w_c(t)T(t)}{h(t)} + \frac{1}{\rho A}\frac{w_h(t)T_h(t)}{h\left(t\right)} + \frac{1}{\rho A}\frac{w_c(t)T_c(t)}{h(t)}$$

- Output equation

$$\boldsymbol{y} = \boldsymbol{g}\left(\boldsymbol{x}(t),\boldsymbol{u}(t)\right) = \begin{bmatrix} g_1\left(\mathrm{x}_1(t),\mathrm{x}_2(t),\mathrm{u}_1(t),\mathrm{u}_2(t),\mathrm{u}_3(t),\mathrm{u}_4(t)\right) \\ g_2\left(\mathrm{x}_1(t),\mathrm{x}_2(t),\mathrm{u}_1(t),\mathrm{u}_2(t),\mathrm{u}_3(t),\mathrm{u}_4(t)\right) \end{bmatrix}$$

where

$$g_1(\boldsymbol{x},\boldsymbol{u}) = h(t)$$
$$g_2(\boldsymbol{x},\boldsymbol{u}) = T(t)$$

- Summary – Nonlinear Model
  - State equations

$$\begin{cases} \dfrac{d}{dt}h(\mathrm{t}) = \dfrac{1}{\rho A}w_h(\mathrm{t}) + \dfrac{1}{\rho A}w_c(\mathrm{t}) - \dfrac{1}{\rho AR}\sqrt{h(t)} \\ \dfrac{d}{dt}T(t) = -\dfrac{1}{\rho A}\dfrac{w_h(t)T(t)}{h(t)} - \dfrac{1}{\rho A}\dfrac{w_c(t)T(t)}{h(t)} + \dfrac{1}{\rho A}\dfrac{w_h(t)T_h(t)}{h(t)} + \dfrac{1}{\rho A}\dfrac{w_c(t)T_c(t)}{h(t)} \end{cases}$$

  - Output equations

$$\begin{cases} y_1(t) = h(\mathrm{t}) \\ y_2(t) = T(t) \end{cases}$$

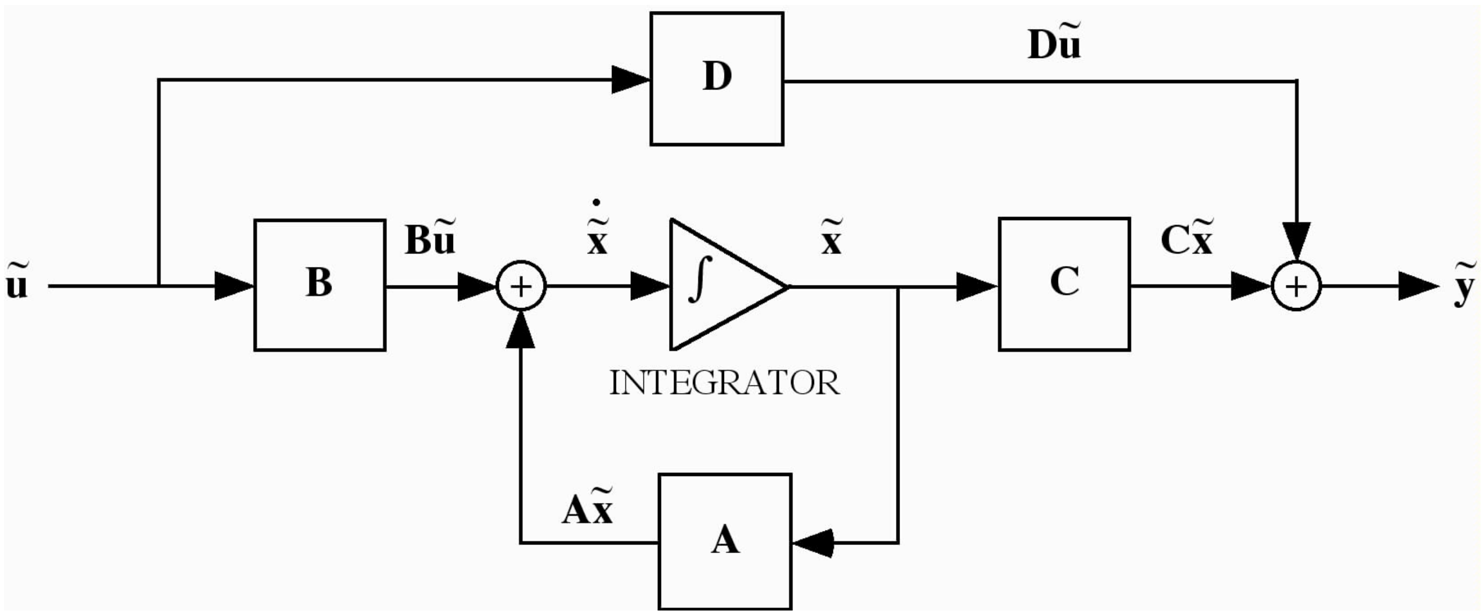
- **Linearization about a steady-state operating point**
  - A standard procedure can be used to linearize the resulting model
  - State equations

$$
\begin{bmatrix} \dot{\tilde{h}}(t) \\ \dot{\tilde{T}}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2\rho A R\sqrt{\bar{h}}} & 0 \\ -\dfrac{\bar{w}_h \bar{T}_h + \bar{w}_c \bar{T}_c - \left(\bar{w}_h + \bar{w}_c\right)\bar{T}}{\rho A \bar{h}^2} & -\dfrac{\bar{w}_h + \bar{w}_c}{\rho A \bar{h}} \end{bmatrix} \begin{bmatrix} \tilde{h}(t) \\ \tilde{T}(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} \dfrac{1}{\rho A} & \dfrac{1}{\rho A} & 0 & 0 \\ \dfrac{\bar{T}_h - \bar{T}}{\rho A \bar{h}} & \dfrac{\bar{T}_c - \bar{T}}{\rho A \bar{h}} & \dfrac{\bar{w}_h}{\rho A \bar{h}} & \dfrac{\bar{w}_c}{\rho A \bar{h}} \end{bmatrix} \begin{bmatrix} \tilde{w}_h(t) \\ \tilde{w}_c(t) \\ \tilde{T}_h(t) \\ \tilde{T}_c(t) \end{bmatrix}
$$

- Output equations

$$\begin{bmatrix}\tilde{y}_1(\mathrm{t})\\ \tilde{y}_2(\mathrm{t})\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{h}(\mathrm{t})\\ \tilde{T}(\mathrm{t})\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{w}_h(\mathrm{t})\\ \tilde{w}_c(\mathrm{t})\\ \tilde{T}_h(\mathrm{t})\\ \tilde{T}_c(\mathrm{t})\end{bmatrix}$$



(II) REVIEW OF LINEAR ALGEBRA

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Real and Complex Numbers

- Scalars
  - $\mathbb{R}$ Set of real numbers
  - $\mathbb{C} = \{ \sigma + i\,\omega : \sigma,\ \omega \in \mathbb{R} \}$ Set of complex numbers
  - $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ Set of integers
  - $\mathbb{N} = \{ 0, 1, 2, \ldots \}$ Set of natural numbers
- Vectors
  - Set of ordered n-tuples (Cartesian product space)

$$\mathbb{R}^n = \left\{ \{x_1, x_2, x_3, \cdots, x_n\} : x_i \in \mathbb{R},\ \ i = 1, 2, \ldots, n \right\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

$$\boldsymbol{x} \in \mathbb{R}^n \quad \Rightarrow \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \boldsymbol{x} \text{ is a vector in } \mathbb{R}^n$$

Linear Combination of Vectors

- Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be a set of (real or complex) scalars
- **Definition - Linear combination of vectors**

$$\alpha_1\, \boldsymbol{x}_1 + \alpha_2\, \boldsymbol{x}_2 + \alpha_3\, \boldsymbol{x}_1 + \cdots + \alpha_n\, \boldsymbol{x}_n$$

- **Definition - Linear independence of vectors**. A set of vectors

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$$

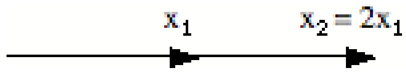
is said to be ***linearly independent*** if the only solution to the equation

$$\alpha_1\, \boldsymbol{x}_1 + \alpha_2\, \boldsymbol{x}_2 + \alpha_3\, \boldsymbol{x}_1 + \cdots + \alpha_n\, \boldsymbol{x}_n = \boldsymbol{0}$$

is the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$.

Otherwise, the vectors are said to be *linearly dependent*.

- Two collinear vectors are linearly dependent



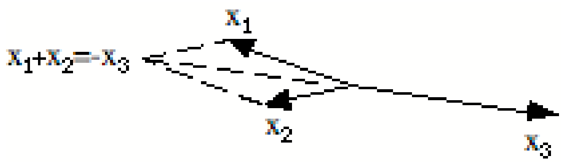
$\alpha 1 = 2,\ \alpha 2 = -1 \ \Rightarrow 2\mathbf{x}1 - 1\mathbf{x}2 = \mathbf{0}$

- Two noncollinear vectors are linearly independent

only $\alpha_1 = \alpha_2 = 0$ can make

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 = \mathbf{0}$$

- Three noncollinear vectors lying on the same plane are linearly dependent



$\alpha 1 = 1,\ \alpha 2 = 1,\ \alpha 3 = 1 \ \Rightarrow 1\mathbf{x}1 + 1\mathbf{x}2 +$

$1\mathbf{x}3 = \mathbf{0}$

Matrices

- An m x n matrix is an array comprised of m rows and n columns

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n-1} & a_{m,n} \end{bmatrix} = \left(a_{ji}\right)$$

- **Real matrix**: $\boldsymbol{A} \in \mathbb{R}^{n \times m}$ is such that $a_{ij} \in \mathbb{R} \ \forall (i, j)$
- **Complex matrix:** $\boldsymbol{A} \in \mathbb{C}^{n \times m}$ is such that $a_{ij} \in \mathbb{C} \ \forall (i, j)$

The succinct notation $\mathbf{A} = (a_{ij})$ or $\mathbf{A} = [(a_{ij})]$ is often used.

- A matrix as a collection of column vectors

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_\mathrm{n} \end{bmatrix} \in \mathbb{C}^{m \times n}$$

where

$$\boldsymbol{a}_1 = \begin{bmatrix} \mathrm{a}_{11} \\ \mathrm{a}_{21} \\ \vdots \\ \mathrm{a}_{\mathrm{m}1} \end{bmatrix} \in \mathbb{C}^m \quad \boldsymbol{a}_2 = \begin{bmatrix} \mathrm{a}_{12} \\ \mathrm{a}_{22} \\ \vdots \\ \mathrm{a}_{\mathrm{m}2} \end{bmatrix} \in \mathbb{C}^m \quad \boldsymbol{a}_\mathrm{i} = \begin{bmatrix} \mathrm{a}_{1\mathrm{i}} \\ \mathrm{a}_{2\mathrm{i}} \\ \vdots \\ \mathrm{a}_{\mathrm{mi}} \end{bmatrix} \in \mathbb{C}^m \quad \boldsymbol{a}_\mathrm{n} = \begin{bmatrix} \mathrm{a}_{1\mathrm{n}} \\ \mathrm{a}_{2\mathrm{n}} \\ \vdots \\ \mathrm{a}_{\mathrm{mn}} \end{bmatrix} \in \mathbb{C}^m$$

- A matrix as a collection of row vectors

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{b}_1^{\mathrm{T}} \\ \boldsymbol{b}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{b}_{\mathrm{m}}^{\mathrm{T}} \end{bmatrix} \in \mathbb{C}^{m \times n}$$

$$\boldsymbol{b}_1^{\mathrm{T}} = \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1\mathrm{n}} \end{bmatrix} \in \mathbb{C}^{1 \times n}$$

$$\boldsymbol{b}_2^{\mathrm{T}} = \begin{bmatrix} \mathrm{a}_{21} & \mathrm{a}_{22} & \cdots & \mathrm{a}_{2\mathrm{n}} \end{bmatrix} \in \mathbb{C}^{1 \times n}$$

$$\vdots$$

$$\boldsymbol{b}_{\mathrm{m}}^{\mathrm{T}} = \begin{bmatrix} \mathrm{a}_{\mathrm{m}1} & \mathrm{a}_{\mathrm{m}2} & \cdots & \mathrm{a}_{\mathrm{mn}} \end{bmatrix} \in \mathbb{C}^{1 \times n}$$

or

$$\boldsymbol{b}_2 = \begin{bmatrix} \mathrm{a}_{21} \\ \mathrm{a}_{22} \\ \vdots \\ \mathrm{a}_{2\mathrm{n}} \end{bmatrix} \in \mathbb{C}^{n} \quad \boldsymbol{b}_{\mathrm{i}} = \begin{bmatrix} \mathrm{a}_{\mathrm{i}1} \\ \mathrm{a}_{\mathrm{i}2} \\ \vdots \\ \mathrm{a}_{\mathrm{in}} \end{bmatrix} \in \mathbb{C}^{n} \quad \cdots \quad \boldsymbol{b}_{\mathrm{m}} = \begin{bmatrix} \mathrm{a}_{\mathrm{m}1} \\ \mathrm{a}_{\mathrm{m}2} \\ \vdots \\ \mathrm{a}_{\mathrm{mn}} \end{bmatrix} \in \mathbb{C}^{n}$$

- **Partitioning a matrix into submatrices**

$$
A = \left[\begin{array}{cccccc|c}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2,n-1} & a_{2,n} \\
\hline
a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{m,n-1} & a_{m,n}
\end{array}\right] = \left[\begin{array}{c|c}
\boldsymbol{B} & \boldsymbol{C} \\
\hline
\boldsymbol{D} & \boldsymbol{E}
\end{array}\right]
$$

Transposition of a matrix

If $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \in \mathbb{C}^{3\times 2}$ then $\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \in \mathbb{C}^{2\times 3}$

- if $\boldsymbol{A} \in \mathbb{C}^{n\times m}$, then $\boldsymbol{A} \in \mathbb{C}^{m\times n}$

- Properties

$$\left(\boldsymbol{A}+\boldsymbol{B}\right)^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}} \qquad \left(\boldsymbol{AB}\right)^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \qquad \left(\boldsymbol{ABC}\right)^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}$$

$$\left[\begin{array}{c:c} \boldsymbol{B} & \boldsymbol{C} \end{array}\right]^{\mathrm{T}} = \left[\begin{array}{c} \boldsymbol{B}^{\mathrm{T}} \\ \hdashline \boldsymbol{C}^{\mathrm{T}} \end{array}\right] \qquad \left[\begin{array}{c} \boldsymbol{B} \\ \hdashline \boldsymbol{D} \end{array}\right]^{\mathrm{T}} = \left[\begin{array}{c:c} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{D}^{\mathrm{T}} \end{array}\right] \qquad \left[\begin{array}{c:c} \boldsymbol{B} & \boldsymbol{C} \\ \hdashline \boldsymbol{D} & \boldsymbol{E} \end{array}\right]^{\mathrm{T}} = \left[\begin{array}{c:c} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{D}^{\mathrm{T}} \\ \hdashline \boldsymbol{C}^{\mathrm{T}} & \boldsymbol{E}^{\mathrm{T}} \end{array}\right]$$

Determinant of a Square Matrix

- **Special Definitions**

1x1 matrix $\mathbf{A} \in \mathbb{R}^{1x1}$ $\mathbf{A} = (a_{11})$, $det(\mathbf{A}) = a_{11}$

2x2 matrix $\mathbf{A} \in \mathbb{R}^{2x2}$ $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12}$

3x3 matrix $\mathbf{A} \in \mathbb{R}^{3x3}$ $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$det(\mathbf{A}) = a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{21}\, a_{32}\, a_{13}$$
$$- a_{31}\, a_{22}\, a_{13} - a_{21}\, a_{12}\, a_{33} - a_{32}\, a_{23}\, a_{11}$$

- **General Definition**

  - **Preliminary Definition 1**. Minor of element a_{ij}: M_{ij}

    Defined as the determinant of the matrix obtained by eliminating row i and column j from $\boldsymbol{A}$.

  - **Preliminary Definition 2**. Cofactor of element a_{ij}: $A_{ij} = (-1)^{i+j} M_{ij}$

  - **Preliminary Definition 3**. Cofactor of a matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$: $\text{cof}(\boldsymbol{A})$.

$$\text{cof}(\boldsymbol{A}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

Matrix whose (i, j)-th element is the A_{ij} cofactor of element a_{ij}.

- **Definition.** The determinant of a matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is defined as a the following cofactor expansion about <u>any row i or any column j</u> of the matrix $\boldsymbol{A}$:

$$det\left(\boldsymbol{A}\right) := \sum_{k=1}^{n} a_{ik} A_{ik} = \sum_{k=1}^{n} \left(-1\right)^{i+k} a_{ik} M_{ik}$$

or

$$det\left(\boldsymbol{A}\right) := \sum_{k=1}^{n} a_{kj} A_{kj} = \sum_{k=1}^{n} \left(-1\right)^{k+j} a_{kj} M_{kj}$$

- **Remark.** The cofactor expansion of the determinant of a matrix of dimensions nxn consists of n terms, and each one of these terms is the product of n elements. Furthermore, each term contains one and only one element from each row, and one and only one element from each column.

- **Example**

  - Cofactor expansion of matrix $\boldsymbol{A}$ about the first row:

$$\det(\boldsymbol{A}) := \sum_{k=1}^{n} a_{1k} A_{1k} = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} M_{1k}$$

  - Derive the formula for the determinant of the 3x3 matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

    using the cofactor expansion about the first row

    - Minors

$$M_{11} = \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) = a_{22}a_{33} - a_{32}a_{23}$$

$$M_{12} = \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) = a_{21}a_{33} - a_{31}a_{23}$$

$$M_{13} = \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{21}a_{32} - a_{31}a_{22}$$

- Cofactor expansion about the first row

$$\begin{aligned}
\det(\boldsymbol{A}) &= (-1)^{1+1} a_{11}M_{11} + (-1)^{1+2} a_{12}M_{12} + (-1)^{1+3} a_{13}M_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\
&\quad + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}
\end{aligned}$$

Properties of Determinants

Let $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_\mathrm{p} & \mathbf{a}_\mathrm{q} & \cdots & \mathbf{a}_\mathrm{n} \end{bmatrix}$, where $\boldsymbol{a}_i \in \mathbb{R}^n$

(a) Multiplying a column (row) by a scalar is equivalent to multiplying the *det*(**A**) by the scalar

$$\det\left(\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \alpha\boldsymbol{a}_i & \cdots & \boldsymbol{a}_n \end{bmatrix}\right) = \alpha \det\left(\boldsymbol{A}\right), \quad \alpha \in \mathbb{R} \quad \alpha \in \mathbb{R}$$

(b) $\det\left(\alpha\boldsymbol{A}\right) = \alpha^n \det\left(\boldsymbol{A}\right)$

(c) The determinant of a matrix with a column (row) of zeros is identically zero

$$\det\left(\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{a}_n \end{bmatrix}\right) = 0$$

(d) Swapping any two columns (rows) changes the sign of the determinant

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_\mathrm{p} & \mathbf{a}_\mathrm{q} & \cdots & \mathbf{a}_\mathrm{n} \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_\mathrm{q} & \mathbf{a}_\mathrm{p} & \cdots & \mathbf{a}_\mathrm{n} \end{bmatrix}\right)$$

(e) Adding a multiple of a column (row) to another column (row) does not affec $\det(\mathbf{A})$

$$\det\left(\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p & \mathbf{a}_q & \cdots & \mathbf{a}_n\end{bmatrix}\right) = \det\left(\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p & (\mathbf{a}_q + \alpha\mathbf{a}_p) & \cdots & \mathbf{a}_n\end{bmatrix}\right)$$

(f) $\det(\mathbf{A}) = \det(\mathbf{A}^T)$

(g) $\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A})\det(\mathbf{B}), \qquad (\mathbf{A}, \mathbf{B} \text{ square})$

(h) $\det(\mathbf{I}) = 1$

(i) $\det\left(\mathbf{A}^{-1}\right) = \dfrac{1}{\det(\mathbf{A})}$

(j) $\mathit{det}\,(\mathbf{I} + \mathbf{AB}) = \mathit{det}\,(\mathbf{I} + \mathbf{BA})$

Matrix Rank

- **Definition.** Matrix $A \in \mathbb{R}^{n\times m}$ has column-rank α if at most α columns of A are linearly independent
- **Definition.** Matrix $A \in \mathbb{R}^{n\times m}$ has *row-rank* β if at most β rows of **A** are linearly independent
- **Definition.** Matrix $A \in \mathbb{R}^{n\times m}$ has *full column-rank* if $\alpha = m \ (rk(A) = m)$
- **Definition.** Matrix $A \in \mathbb{R}^{n\times m}$ has *full row-rank* if $\beta = n \ (rk(A) = n)$
- **THEOREM.**
  - The row-rank and column-rank of any *m* x *n* matrix are equal.
- **COROLLARY.**
  - $rk(\mathbf{A})$ = number of independent columns of **A**

    = number of independent rows of **A**
- **Remark.** Let $A \in \mathbb{R}^{n\times m}$, then $rk\,(\mathbf{A}) \le min\,(m, n)$

- **THEOREM**. Let $\boldsymbol{A} \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ be an arbitrary matrix, and let square matrices $\boldsymbol{B} \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}}$ and $\boldsymbol{A} \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ be full-rank matrices. Then

$$rk\,(\mathbf{A}) = rk\,(\mathbf{AB}) = rk\,(\mathbf{CA})$$

Interpretation:

Multiplication by a full-rank matrix does not affect rank

Matrix Inversion

- Inversion is defined only for square matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ (or $\boldsymbol{A} \in \mathbb{C}^{m \times n}$)

- **Definition**. Matrix $\boldsymbol{A}^{-1} \in \mathbb{R}^{n \times n}$ is the inverse of matrix $\boldsymbol{A}$ if and only if it satisfies the equalities

$$\boxed{\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}}$$

- **Definition**. Matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is said to be <u>nonsingular</u> if there exists a matrix-inverse $\boldsymbol{A}^{-1} \in \mathbb{R}^{n \times n}$. Otherwise, $\boldsymbol{A}$ is said to be <u>singular</u>.

  - *Remark*: The matrix inverse is also square.

- **THEOREM**. Existence of a matrix inverse for a square matrix $A \in \mathbb{R}^{n\times n}$. The following statements are equivalent:

    - (i) Matrix $A \in \mathbb{R}^{n\times n}$ is nonsigular if and only if A is full rank.

    - (ii) Matrix $A \in \mathbb{R}^{n\times n}$ is nonsigular if and only if $rk(A) = n$.

    - (iii) Matrix $A \in \mathbb{R}^{n\times n}$ is nonsigular if and only if $\det(A) \neq 0$.

- Properties of the matrix-inversion operation

    - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    - $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

Left and Right Inverses

- Nonsquare (rectangular) matrices have no inverse. However, they may have a left or a right inverse

- **THEOREM**. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a full-column rank, then the square matrix $\boldsymbol{A}^{\mathrm{T}} \boldsymbol{A} \in \mathbb{R}^{n \times n}$ is full rank.

  Equivalently: $\operatorname{rk}(\boldsymbol{A}) = n \quad \Rightarrow \quad \operatorname{rk}(\boldsymbol{A}^T \boldsymbol{A}) = n$

- **THEOREM**. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a full-row rank, then the square matrix $\boldsymbol{A}\boldsymbol{A}^T \in \mathbb{R}^{\mathrm{m} \times \mathrm{m}}$ is full rank.

  Equivalently: $\operatorname{rk}(\boldsymbol{A}) = m \quad \Rightarrow \quad \operatorname{rk}(\boldsymbol{A}^T \boldsymbol{A}) = m$

- **Definition**. $A^{-L} \in \mathbb{R}^{n \times m}$ is the left inverse of $A \in \mathbb{R}^{m \times n}$ if and only if

$$A^{-L} A = I .$$

- **Definition**. $A^{-R} \in \mathbb{R}^{n \times m}$ is the right inverse of $A \in \mathbb{R}^{m \times n}$ if and only if

$$A A^{-R} = I .$$

- **THEOREM**. There exists a left inverse of $A \in \mathbb{R}^{m \times n}$ if and only if A is full-column rank. Furthermore, the left inverse is <u>unique</u> and given by

$$A^{-L} = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} \in \mathbb{R}^{n \times m}.$$

  - *Remark*: $A^{-L} A = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} A = I \in \mathbb{R}^{n \times n}$

- **THEOREM**. There exists a right inverse of $A \in \mathbb{R}^{m \times n}$ if and only if A is full-row rank. Furthermore, the right inverse is <u>not unique</u>, and one right-inverse is given by

$$A^{-R} = A^{\mathrm{T}} (A A^{\mathrm{T}})^{-1} \in \mathbb{R}^{n \times m}.$$

  - *Remark*: $A A^{-R} = A A^{\mathrm{T}} (A A^{\mathrm{T}})^{-1} = I \in \mathbb{R}^{m \times m}$

The Adjoint and Inverse Matrices

- **Definition**. The adjoint of a square matrix is defined as the transpose of the matrix of cofactors:

$$adj(\boldsymbol{A}) = \left[\; cof(\boldsymbol{A}) \;\right]^{T}$$

  - Remark:

$$adj(\boldsymbol{A}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^{T} = \begin{bmatrix} (-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & \cdots & (-1)^{1+n}M_{1n} \\ (-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & \cdots & (-1)^{2+n}M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & \cdots & (-1)^{n+n}M_{nn} \end{bmatrix}^{T}$$

- **THEOREM.**

$$\boxed{A^{-1} = \frac{1}{det(A)}\, adj(A)}$$

- *Remark*:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & \cdots & (-1)^{1+n}M_{1n} \\ (-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & \cdots & (-1)^{2+n}M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & \cdots & (-1)^{n+n}M_{nn} \end{bmatrix}^T$$

Inverse of a 2x2 Matrix

$$\text{Let } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

- Procedure to find the inverse (assuming that $det(\boldsymbol{A}) \neq 0$):

  - Find the cofactors for each of the elements of the matrix:

$$A_{11} = (-1)^{1+1}\det(d) = d, \quad A_{12} = (-1)^{1+2}\det(c) = -c$$

$$A_{21} = (-1)^{2+1}\det(b) = -b, A_{22} = (-1)^{2+2}\det(a) = a$$

- Construct the adjoint matrix and find the determinant

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(\mathbf{A}) = ad - cb$$

- Construct the inverse matrix using the adjoint theorem

$$\boxed{\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}) = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Eigenvalues and Eigenvectors

- Let $A \in \mathbb{R}^{n\times n}$ and consider the equation

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{1}$$

- **Definition**. Every nonzero vector $v \in \mathbb{C}^n$ that satisfies (1) is an **eigenvector** of A, and every scalar $\lambda \in \mathbb{C}$ that satisfies (1) is an **eigenvalue** of A.
- *Notation*:

Characteristic matrix: $\lambda\mathbf{I} - \mathbf{A}$

Characteristic equation: $det(\lambda I - A) = 0$

Characteristic polynomial:

$$\begin{aligned} det(\lambda I - A) &= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{n-1})(\lambda - \lambda_n) \end{aligned}$$

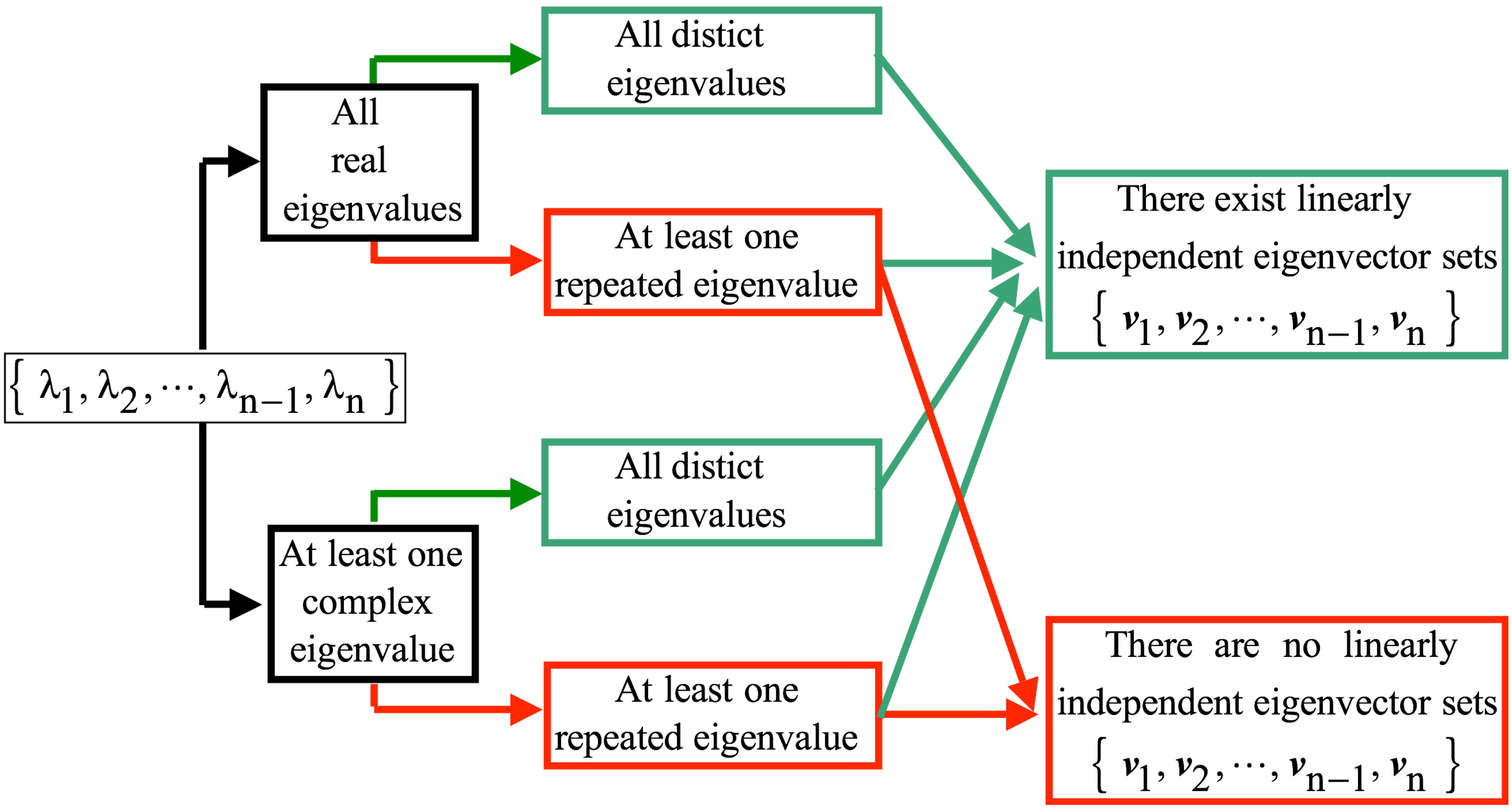
- **THEOREM**. The eigenvalues of $\boldsymbol{A}$ are the roots of the characteristic equation.

- Observations:
  - There is a unique set of *n* eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n\}$
  - Each eigenvalue satisfies $\lambda_i^{\,n} + a_{n-1}\lambda_i^{\,n-1} + \cdots + a_1\lambda_i + a_0 = 0$
  - An eigenvalue can be zero, but no eigenvector can be zero
  - Given that A is a real matrix, then all complex eigenvalues appear as a complex-conjugate pair in the set $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n\}$
  - The set of eigenvectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n\}$ is not unique.

- **Definition**. The of <u>spectrum</u> of matrix $\boldsymbol{A}$ is the set of eigenvalues, and is denoted as

$$\Lambda(\boldsymbol{A}) = \left\{ \{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n\} \ : \ \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \right\}$$

Classification of Eigenvalues



(III) THE MATRIX EXPONENTIAL FUNCTION

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Contents

Canonical Matrix Forms

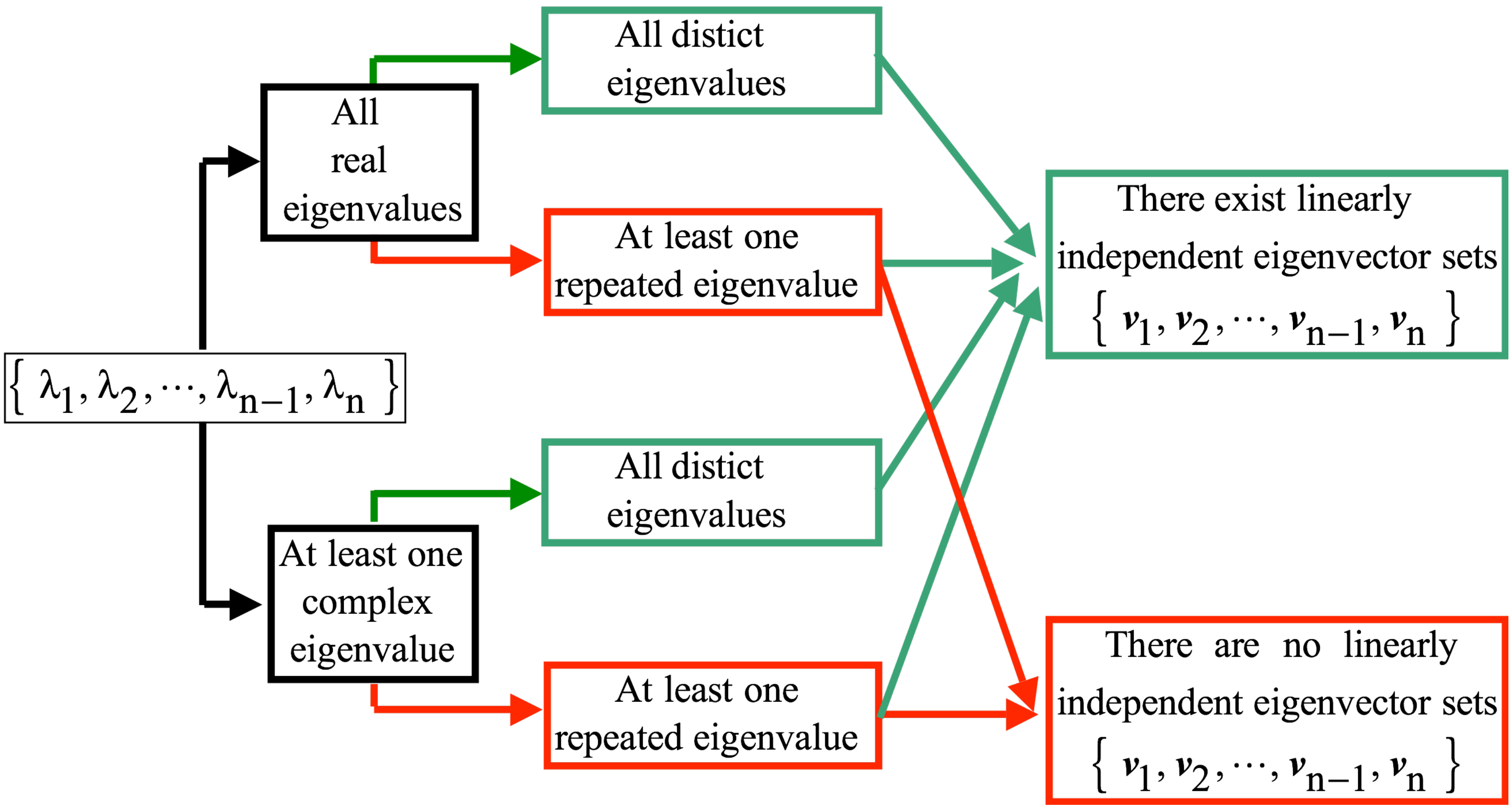
- Spectrum and eigenvalue sets of matrix $\boldsymbol{A} \in \Re^{nxn}$

  $\Lambda(\boldsymbol{A}) = \{ \lambda_1, \lambda_2, \ldots , \lambda_{n-1}, \lambda_n \}$

  $\mathcal{V}(\boldsymbol{A}) = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots , \boldsymbol{v}_{n-1} , \boldsymbol{v}_n \}$

- Canonical forms for matrix $\boldsymbol{A}$
  - Diagonal canonical form $\boldsymbol{D}$: $\boldsymbol{D} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}$
  - Jordan canonical form $\boldsymbol{J}$: $\boldsymbol{J} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}$
  - Square canonical form $\boldsymbol{K}$: $\boldsymbol{K} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}$
  - Square-Jordan canonical form $\boldsymbol{Q}$: $\boldsymbol{Q} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}$

Classification of Eigenvalues



Eigenvalue Multiplicity

- Algebraic multiplicity m_i

  - $\Lambda(\boldsymbol{A}) = \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \} = \{ 2, -4, -4, 5, 5, 5 \}$

$\lambda_1(\boldsymbol{A}) = 2$ algebraic multiplicity $m_1 = 1$ (*distinct*)

$\lambda_2(\boldsymbol{A}) = -4$ algebraic multiplicity $m_2 = 2$ (*repeated*)

$\lambda_4(\boldsymbol{A}) = 5$ algebraic multiplicity $m_3 = 3$ (*repeated*)

- Eigenvector sets for repeated eigenvalues

$$\Lambda(\boldsymbol{A}) = \{\ \lambda_1, \lambda_2, \lambda_3, \lambda_4\ \lambda_5, \lambda_6\ \} = \{\ \lambda_1, \lambda_2, \lambda_2,\ \lambda_4,\ \lambda_4,\ \lambda_4\ \}$$

$$\mathbb{V}(\boldsymbol{A}) = \{\ \boldsymbol{v}_1, \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4,\ \boldsymbol{v}_5, \boldsymbol{v}_6\ \} = \{\ \boldsymbol{v}_1, \boldsymbol{v}_{21}, \boldsymbol{v}_{22}, \boldsymbol{v}_{41}, \boldsymbol{v}_{42}, \boldsymbol{v}_{43}\ \}$$

where λ_1, λ_2 and λ_3, are ***distinct***

- Eigenvalue set associated with each distinct eigenvalue

$$m_1 = m(\lambda_1) = 1 \qquad \mathbb{V}(\lambda_1) = \{\ \boldsymbol{v}_1\ \}$$

$$m_2 = m(\lambda_2) = 2 \qquad \mathbb{V}(\lambda_2) = \{\ \boldsymbol{v}_1, \boldsymbol{v}_2\ \} = \{\ \boldsymbol{v}_{21}, \boldsymbol{v}_{22}\ \}$$

$$m_4 = m(\lambda_4) = 3 \qquad \mathbb{V}(\lambda_4) = \{\ \boldsymbol{v}_4,\ \boldsymbol{v}_5, \boldsymbol{v}_6\ \} = \{\ \boldsymbol{v}_{41}, \boldsymbol{v}_{42}, \boldsymbol{v}_{43}\ \}$$

- Geometric multiplicity $\boldsymbol{g_i}$ of an eigenvalue – Example

$$m_4 = m(\lambda_4) = 3 \qquad \mathbb{V}(\lambda_4) = \{\ \boldsymbol{v}_4,\ \boldsymbol{v}_5, \boldsymbol{v}_6\ \} = \{\ \boldsymbol{v}_{41}, \boldsymbol{v}_{42}, \boldsymbol{v}_{43}\ \}$$

- $g_4 = 1$ if $\mathbb{V}(\lambda_4)$ has only 1 independent vector
- $g_4 = 2$ if $\mathbb{V}(\lambda_4)$ has only 2 independent vectors
- $g_4 = 3$ if $\mathbb{V}(\lambda_4)$ has 3 independent vectors

- Example of case of algebraic multiplicity equal to 2

$$\Lambda(\boldsymbol{A}) = \{\ \lambda_1, \lambda_2\ \} = \{\ -3, -3\ \} \qquad \mathbb{V}(\boldsymbol{A}) = \{\ \boldsymbol{v}_1, \boldsymbol{v}_2\ \}$$

$$\lambda_1(\boldsymbol{A}) = -3 \quad m_1 = 2 \ \ (\textit{repeated})$$

- If $\{\ \boldsymbol{v}_1, \boldsymbol{v}_2\ \}$ is a linearly dependent set $\Rightarrow$ $\boxed{g_i = 1}$
- If $\{\ \boldsymbol{v}_1, \boldsymbol{v}_2\ \}$ is a linearly independent set $\Rightarrow$ $\boxed{g_i = 2}$

- Eigenvectors and principal vectors
  - Case of a repeated eigenvalue $\lambda_1(\boldsymbol{A})$
    - Algebraic multiplicity $m_1 = m(\lambda_1) = 3$

- Geometric multiplicity $g_1 = m(\lambda_1) = 1$
- Eigenvector equations

$$\boxed{\boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{v}_1\lambda_1, \qquad \boldsymbol{A}\boldsymbol{v}_2 = \boldsymbol{v}_2\lambda_2, \qquad \boldsymbol{A}\boldsymbol{v}_3 = \boldsymbol{v}_3\lambda_3}$$

- Eigenvector set $\mathbb{V}(\lambda_1) = \left\{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \boldsymbol{v}_3 \right\}$ is NOT linearly independent
- Principal-vector equations. Define $\boldsymbol{w}_1 = \boldsymbol{v}_1$; then

$$\boxed{\boldsymbol{A}\boldsymbol{w}_1 = \boldsymbol{w}_1\lambda_1, \qquad \boldsymbol{A}\boldsymbol{w}_2 = \lambda_1\boldsymbol{w}_2 + \boldsymbol{w}_1, \quad \boldsymbol{A}\boldsymbol{w}_3 = \lambda_1\boldsymbol{w}_3 + \boldsymbol{w}_2}$$

- Generalized eigenvector set $\mathbb{W}(\lambda_1) = \left\{ \boldsymbol{v}_1, \ \mathrm{w}_2, \mathrm{w}_3 \right\}$
- **THEOREM**. $\mathbb{W}(\lambda_1) = \left\{ \boldsymbol{v}_1, \ \mathrm{w}_2, \mathrm{w}_3 \right\}$ is linearly independent

Similarity Transformations

- **Definition**. Two square matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{R} \in \mathbb{R}^{n \times n}$ are said to be similar if there exists a **nonsingular** matrix $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ such that

$$\boldsymbol{R} = \boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}$$

- Similarity transformation: the operation $\boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}$

- Properties of similarity transformations

$$\lambda\left(\boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}\right) = \lambda\left(\boldsymbol{A}\right)$$

$$\operatorname{rk}\left(\boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}\right) = \operatorname{rk}\left(\boldsymbol{A}\right)$$

$$\det\left(\boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}\right) = \det \boldsymbol{A}$$

Diagonal Canonical Form

- Let $A \in \mathbb{R}^{4\times 4}$ have an eigenvalue set $\Lambda(A) = \{\ \lambda_1, \lambda_2, \lambda_3, \lambda_4\ \}$ where all the eigenvalues are (*i*) **real**, and (*ii*) **distinct**.

- Let $\mathbb{V}(A) = \{\ \boldsymbol{v}_1, \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4\ \}$ be an EIGENVECTOR set

- **THEOREM**. The eigenvector set $\mathbb{V}(A)$ is linearly independent

- **Definition**. Modal matrix for $\boldsymbol{A}$

$$\boldsymbol{H} := \left[\ \boldsymbol{v}_1, \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4\ \right] \in \mathbb{R}^{4\times 4}$$

- **COROLARY** The inverse modal matrix $\boldsymbol{H}^{-1} \in \mathbb{R}^{4\times 4}$ exists.

- **THEOREM.**

$$\boxed{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \mathrm{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right) \in \mathbb{R}^{4\times 4}}$$

Jordan Canonical From

- Let $A \in \mathbb{R}^{4\times 4}$ have an eigenvalue set

$$\Lambda(A) = \{\ \lambda_1, \lambda_2, \lambda_3, \lambda_4\ \} = \{\ \lambda_1, \lambda_1, \lambda_1, \lambda_1\ \}$$

where all the eigenvalues are (*i*) **real**, and (*ii*) **repeated**.

- Let $\mathbb{V}(A) = \{\ \boldsymbol{v}_1, \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4\ \}$ be an EIGENVECTOR set
    - Linearly dependent

- Let $\mathbb{W}(A) = \{\ \boldsymbol{w}_1, \boldsymbol{w}_2,\ \boldsymbol{w}_3,\ \boldsymbol{w}_4\ \}$ be a PRINCIPAL VECTOR set, where

$$\boldsymbol{A}\boldsymbol{w}_1 = \boldsymbol{w}_1\lambda_1, \qquad \boldsymbol{A}\boldsymbol{w}_2 = \lambda_1\boldsymbol{w}_2 + \boldsymbol{w}_1, \quad \boldsymbol{A}\boldsymbol{w}_3 = \lambda_1\boldsymbol{w}_3 + \boldsymbol{w}_2, \quad \boldsymbol{A}\boldsymbol{w}_4 = \lambda_1\boldsymbol{w}_4 + \boldsymbol{w}_3$$

- **THEOREM**. The set $\mathbb{W}(A)$ is linearly independent

- **Definition**. Modal matrix for $\boldsymbol{A}$

$$\boldsymbol{H} := \left[\, \boldsymbol{w}_1,\ \boldsymbol{w}_2,\ \ \boldsymbol{w}_3,\ \ \boldsymbol{w}_4 \,\right] \in \mathbb{R}^{4\times 4}$$

- **COROLARY** The inverse modal matrix $\boldsymbol{H}^{-1} \in \mathbb{R}^{4\times 4}$ exists.

- **THEOREM**.

$$\boxed{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \boldsymbol{J} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \in \mathbb{R}^{4\times 4}}$$

Square Canonical Form

- Let $A \in \mathbb{R}^{4\times 4}$ have an eigenvalue set

  $\Lambda(A) = \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} = \{ \sigma_1 + j\omega_1, \sigma_1 - j\omega_1, \sigma_2 + j\omega_2, \sigma_2 - j\omega_2 \}$

  where all the eigenvalues are (*i*) **complex**, and (*ii*) **distinct**.

- Let $\mathbb{V}(A) = \{ \boldsymbol{v}_1, \bar{\boldsymbol{v}}_1, \boldsymbol{v}_2, \bar{\boldsymbol{v}}_2 \}$ be an EIGENVECTOR set , where

$$\boldsymbol{v}_1 = \mathrm{Re}(\boldsymbol{v}_1) + j\ \mathrm{Im}(\boldsymbol{v}_1) \qquad \bar{\boldsymbol{v}}_1 = \mathrm{Re}(\boldsymbol{v}_1) - j\ \mathrm{Im}(\boldsymbol{v}_1)$$

$$\boldsymbol{v}_2 = \mathrm{Re}(\boldsymbol{v}_2) + j\ \mathrm{Im}(\boldsymbol{v}_2) \qquad \bar{\boldsymbol{v}}_2 = \mathrm{Re}(\boldsymbol{v}_2) - j\ \mathrm{Im}(\boldsymbol{v}_2)$$

  - Linearly independent but complex

- Define $\mathbb{V}'(A) = \{ \mathrm{Re}(\boldsymbol{v}_1), \mathrm{Im}(\boldsymbol{v}_1), \mathrm{Re}(\boldsymbol{v}_2), \mathrm{Im}(\boldsymbol{v}_2) \}$
  - No complex elements

- **THEOREM**. The set $\mathbb{V}'(A)$ is linearly independent

- **Definition**. Modal matrix for $\boldsymbol{A}$

$$\boldsymbol{H} := \left[\ \mathrm{Re}(\boldsymbol{v}_1)\ ,\ \mathrm{Im}(\boldsymbol{v}_1),\ \ \mathrm{Re}(\boldsymbol{v}_2),\ \ \mathrm{Im}(\boldsymbol{v}_2)\ \right] \in \mathbb{R}^{4\times 4}$$

- **COROLARY** The inverse modal matrix $\boldsymbol{H}^{-1} \in \mathbb{R}^{4\times 4}$ exists.

- **THEOREM**.

$$\boxed{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \boldsymbol{K} = \left[\begin{array}{cc|cc} \sigma_1 & \omega_1 & 0 & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 \\ \hline 0 & 0 & \sigma_2 & \omega_2 \\ 0 & 0 & -\omega_2 & \sigma_2 \end{array}\right] \in \mathbb{R}^{4\times 4}}$$

Square-Jordan Canonical Form

- Let $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ have an eigenvalue set

$$\Lambda(\boldsymbol{A}) = \{\ \lambda_1, \lambda_2, \lambda_3, \lambda_4\ \} = \{\ \sigma + j\omega, \sigma - j\omega, \sigma + j\omega, \sigma - j\omega\}$$

where the eigenvalues are (*i*) **<u>complex</u>**, and (*ii*) **<u>repeated</u>**.

- Let $\mathbb{V}(\boldsymbol{A}) = \{\ \boldsymbol{v}, \bar{\boldsymbol{v}},\ \boldsymbol{v},\ \bar{\boldsymbol{v}}\ \}$ be an EIGENVECTOR set , where

$$\boldsymbol{v} = \ \operatorname{Re}(\boldsymbol{v}) + j\ \operatorname{Im}(\boldsymbol{v}) \qquad \bar{\boldsymbol{v}} = \operatorname{Re}(\boldsymbol{v}) - j\ \operatorname{Im}(\boldsymbol{v})$$

  - Linearly dependent

- Let $\mathbb{W}(\boldsymbol{A}) = \{\ \boldsymbol{w}_1, \bar{\boldsymbol{w}}_1,\ \bar{\boldsymbol{w}}_2,\ \boldsymbol{w}_2\ \}$ be a PRINCIPAL VECTOR set, where

$$\boldsymbol{w}_1 = \boldsymbol{v}_1, \qquad \boldsymbol{A}\boldsymbol{w}_2 = \lambda \boldsymbol{w}_2 + \boldsymbol{w}_1$$

  - Linearly independent but complex

- Define $\mathbb{W}'(\boldsymbol{A}) = \left\{ \ \mathrm{Re}(\boldsymbol{w}_1)\ , \ \mathrm{Im}(\boldsymbol{w}_1), \ \ \mathrm{Re}(\boldsymbol{w}_2), \ \ \mathrm{Im}(\boldsymbol{w}_2) \ \right\}$
  - No complex elements

- **THEOREM**. The set $\mathbb{W}'(\boldsymbol{A})$ is linearly independent
- **Definition**. Modal matrix for $\boldsymbol{A}$

$$\boldsymbol{H} := \left[\ \boldsymbol{w}_1, \ \boldsymbol{w}_2, \ \ \boldsymbol{w}_3, \ \ \boldsymbol{w}_4 \ \right] \in \mathbb{R}^{4\times 4}$$

- Define $\mathbb{V}'(\boldsymbol{A}) = \left\{ \ \mathrm{Re}(\boldsymbol{v}_1)\ , \ \mathrm{Im}(\boldsymbol{v}_1), \ \ \mathrm{Re}(\boldsymbol{v}_2), \ \ \mathrm{Im}(\boldsymbol{v}_2) \ \right\}$

- **THEOREM**. The set $\mathbb{V}'(\boldsymbol{A})$ is linearly independent

- **Definition**. Modal matrix for $\boldsymbol{A}$

$$\boldsymbol{H} := \left[\ \mathrm{Re}(\boldsymbol{w}_1)\ ,\ \mathrm{Im}(\boldsymbol{w}_1),\ \ \mathrm{Re}(\boldsymbol{w}_2),\ \ \mathrm{Im}(\boldsymbol{w}_2)\right] \in \mathbb{R}^{4\times 4}$$

- **COROLARY** The inverse modal matrix $\boldsymbol{H}^{-1} \in \mathbb{R}^{4\times 4}$ exists.

- **THEOREM**.

$$\boxed{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \boldsymbol{Q} = \left[\begin{array}{cc|cc} \sigma & \omega & 1 & 0 \\ \omega & \sigma & 0 & 1 \\ \hline 0 & 0 & \sigma & \omega \\ 0 & 0 & \omega & \sigma \end{array}\right] \in \mathbb{R}^{4\times 4}}$$

Analysis of an Example

- The 8x8 real matrix $\boldsymbol{A}$ has a modal matrix $\boldsymbol{H}$ such that

$$\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_{3x3} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{2x2} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_{1x1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_{2x2} \end{bmatrix}$$

- Analyze the nature of the eigenvalues of $\boldsymbol{A}$
  - $\lambda = \lambda_1$ Real ___ or complex ___ $m_1 =$ ___ $g_1 =$ ___

Comment on the eigenvectors and principal vectors ________

____

  - $\lambda = \lambda_4$ Real ___ or complex ___ $m_4 =$ ___ $g_4 =$ ___

    Comment on the eigenvectors and principal vectors ________

    ____

  - $\lambda = \lambda_6$ Real ___ or complex ___ $m_6 =$ ___ $g_6 =$ ___

    Comment on the eigenvectors and principal vectors ________

    ____

  - $\lambda = \lambda_7$ Real ___ or complex ___ $m_6 =$ ___ $g_6 =$ ___

    Comment on the eigenvectors and principal vectors ________

    ____

The Scalar Exponential Function

- Let $\alpha \in \Re$ be a finite number and let $t \in \Re$.
- **Definition**. $$e^{\alpha t} := 1 + t\alpha + \frac{t^2}{2!}\alpha^2 + \frac{t^3}{3!}\alpha^3 + \cdots + \frac{t^k}{k!}\alpha^k + \cdots$$
- **Convergence Property**
  - The power series is uniformly convergent for any finite value of *t* because it satisfies
    $$\lim_{k \to \infty} \frac{t^k}{k!}\alpha^k = 0$$
  - *Reason*: The factorial $k!$ is much greater than $t^k\alpha^k$ at large values of k
- Laplace Transform: $\mathrm{L}\left\{e^{\alpha t}\right\} := \int_0^{+\infty} e^{\alpha t} e^{-st}\, dt = \frac{1}{s - \alpha} = (s - \alpha)^{-1}$

The Matrix Exponential Function

- Let $\boldsymbol{A} \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ and $t \in \Re$.
- **Definition**.

$$e^{\boldsymbol{A}t} = 1 + t\,\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots + \frac{t^k}{k!}\boldsymbol{A}^k + \cdots$$

- **Convergence Property**
  - The power series is uniformly convergent for any finite value of t because it satisfies

$$\lim_{k \to \infty} \frac{t^k}{k!}\boldsymbol{A}^k = \boldsymbol{0}$$

  - *Reason*: The factorial $k!$ is much greater than $t^k b_{ij}$ at large values of k, where b_{ij} is the (i, j)-th element of the matrix $\boldsymbol{A}^k$.

- Properties of the matrix exponential that are **analogous** to those of the scalar exponential function
  - $\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right) = \boldsymbol{A}\, e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$ Differentiation
  - $\frac{d}{dt}\left(e^{\boldsymbol{A}t}\boldsymbol{x}(\mathrm{t})\right) = e^{\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(\mathrm{t}) + e^{\boldsymbol{A}t}\frac{d\boldsymbol{x}}{dt}$ Chain rule
  - $(e^{\boldsymbol{A}t})^T = e^{\boldsymbol{A}^T t}$ Transposition
  - $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{(\boldsymbol{B}+\boldsymbol{A})t}$
  - $e^{\boldsymbol{A}t}e^{\boldsymbol{A}s} = e^{\boldsymbol{A}(t+s)}$
  - $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$ Matrix inverse (nonsingularity)
  - $\mathrm{L}\left\{e^{\boldsymbol{A}t}\right\} := \int_0^{+\infty} e^{\boldsymbol{A}t}e^{-st}\,dt = \left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}$ Laplace Transform

- Properties of the matrix exponential that are **not analogous** to those of the scalar exponential function
  - In general
    - $e^{\boldsymbol{A}t}\, e^{\boldsymbol{B}t} \neq e^{\boldsymbol{B}t}\, e^{\boldsymbol{A}t}$ (Note that $e^{\alpha t}\, e^{\beta t} = e^{\beta t}\, e^{\alpha t}$)
    - $e^{\boldsymbol{A}t} e^{\boldsymbol{B}t} \neq e^{(\boldsymbol{A}+\boldsymbol{B})t}$ (Note that $e^{\alpha t}\, e^{\beta t} = e^{(\alpha+\beta)t}$)
    - *Reason*: In general, matrix multiplication is not commutative with respect to multiplication (i.e., in general $\boldsymbol{YZ} \neq \boldsymbol{ZY}$)
- Effect of Similarity Transformations
    - $e^{\boldsymbol{H}^{-1}\boldsymbol{A}\,\boldsymbol{H}\,t} = \boldsymbol{H}^{-1} e^{\boldsymbol{A}t}\, \boldsymbol{H}$
    - **This is a key relationship used in the formulation of closed-form expressions for the matrix-exponential function**

Closed-Form Expressions

- Matrix $\boldsymbol{A}$ in diagonal canonical form:

$$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \mathbf{D}$$

  - Matrix exponential function for matrix $\boldsymbol{A}$:

$$e^{\boldsymbol{D}\,t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- Matrix $\boldsymbol{A}$ in Jordan canonical form:

$$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_4 \end{bmatrix} = \boldsymbol{J}$$

  - Matrix exponential function for matrix $\boldsymbol{A}$:

$$e^{\boldsymbol{J}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} = \begin{bmatrix} e^{\boldsymbol{J}_1 t} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & e^{\boldsymbol{J}_4 t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & e^{\boldsymbol{J}_6 t} \end{bmatrix}$$

- Matrix $\boldsymbol{A}$ in square canonical form:

$$\boldsymbol{A} = \left[\begin{array}{cc|cc} \sigma_1 & \omega_1 & 0 & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 \\ \hline 0 & 0 & \sigma_2 & \omega_2 \\ 0 & 0 & -\omega_2 & \sigma_2 \end{array}\right] = \boldsymbol{K}$$

- Matrix exponential function for matrix $\boldsymbol{A}$:

$$e^{\boldsymbol{K}\,t} = \left[\begin{array}{cc|cc} e^{\sigma_1 t}\cos\omega_1 t & e^{\sigma_1 t}\sin\omega_1 t & 0 & 0 \\ -e^{\sigma_1 t}\sin\omega_1 t & e^{\sigma_1 t}\cos\omega_1 t & 0 & 0 \\ \hline 0 & 0 & e^{\sigma_2 t}\cos\omega_2 t & e^{\sigma_2 t}\sin\omega_2 t \\ 0 & 0 & -e^{\sigma_2 t}\sin\omega_2 t & e^{\sigma_2 t}\cos\omega_2 t \end{array}\right]$$

- Matrix $\boldsymbol{A}$ in square-Jordan canonical form:

$$\boldsymbol{A} = \left[\begin{array}{cc|cc} \sigma & \omega & 1 & 0 \\ \omega & \sigma & 0 & 1 \\ \hline 0 & 0 & \sigma & \omega \\ 0 & 0 & \omega & \sigma \end{array}\right] = \boldsymbol{Q}$$

  - Matrix exponential function for matrix $\boldsymbol{A}$:

$$e^{\boldsymbol{Q}\,t} = \left[\begin{array}{cc|cc} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t & t\,e^{\sigma t}\cos\omega t & t\,e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t & -t\,e^{\sigma t}\sin\omega t & t\,e^{\sigma t}\cos\omega t \\ \hline 0 & 0 & e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ 0 & 0 & -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{array}\right]$$

Closed Form for an Arbitrary Matrix

- Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and let $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ be a modal matrix for $\boldsymbol{A}$ such that

$$\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix}$$

Then

$$e^{\boldsymbol{A}t} = \boldsymbol{H} \begin{bmatrix} \mathrm{e}^{\boldsymbol{D}t} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{e}^{\boldsymbol{J}t} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathrm{e}^{\boldsymbol{K}t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathrm{e}^{\boldsymbol{Q}t} \end{bmatrix} \boldsymbol{H}^{-1}$$

- Example

$$\boldsymbol{A} = \boldsymbol{H}\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma & \omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega & \sigma \end{bmatrix}\boldsymbol{H}^{-1} = \boldsymbol{H}\begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K} \end{bmatrix}\boldsymbol{H}^{-1}$$

where λ_1, λ_2, λ_3, λ_4, σ, and ω are real numbers.

$$
e^{\boldsymbol{A}t} = \boldsymbol{H}\begin{bmatrix}
e^{\lambda_1 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{\lambda_2 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{\lambda_3 t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & \frac{t^2}{2}e^{\lambda_4 t} & \frac{t^3}{3!}e^{\lambda_4 t} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & \frac{t^2}{2}e^{\lambda_4 t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{\lambda_4 t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\sigma_5 t} cos\, \omega_5 t & e^{\sigma_5 t} sin\, \omega_5 t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{\sigma_5 t} sin\, \omega_5 t & e^{\sigma_5 t} cos\, \omega_5 t
\end{bmatrix}\boldsymbol{H}
$$

(IV) THE LINEAR STATE SPACE SYSTEM

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Contents

Dynamic Systems

$$\Sigma \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) & \textit{(state equation)} \\ \boldsymbol{y}(t) = \boldsymbol{g}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) & \textit{(output equation)} \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & \textit{(initial state)} \end{cases}$$

$\boldsymbol{x} \in \Re^n$	state vector	$\boldsymbol{f} \in \Re^{\mathrm{n}}$	state-derivative map
$\boldsymbol{u} \in \Re^p$	input vector	$\boldsymbol{g} \in \Re^{\mathrm{m}}$	output map
$\boldsymbol{y} \in \Re^m$	output vector	$t \in \Re$	time
$\boldsymbol{x}_0 \in \Re^{\mathrm{n}}$	initial state	$t_0 \in \Re$	initial time

- Order of the dynamic system: n
  - Number of differential equations
- The state and output equations may be **nonlinear**

The Linear State Space System

- **Linear time-varying** (**LTV**) state-space system

$$\Sigma \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) & \textit{(state equation)} \\ \boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) & \textit{(output equation)} \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & \textit{(initial state)} \end{cases}$$

- System matrices are a function of time

$\boldsymbol{A}(t) \in \Re^{\mathrm{n}\times\mathrm{n}}$ state matrix

$\boldsymbol{B}(t) \in \Re^{\mathrm{n}\times\mathrm{p}}$ input matrix

$\boldsymbol{C}(t) \in \Re^{\mathrm{m}\times\mathrm{n}}$ output matrix

$\boldsymbol{D}(t) \in \Re^{\mathrm{m}\times\mathrm{p}}$ feedthrough matrix

- **Linear time-invariant** (**LTI**) state-space system

$$\Sigma \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) & \textit{(state equation)} \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) & \textit{(output equation)} \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & \textit{(initial state)} \end{cases}$$

- Constant system matrices

$\boldsymbol{A} \in \Re^{n\times n}$ state matrix

$\boldsymbol{B} \in \Re^{n\times m}$ input matrix

$\boldsymbol{C} \in \Re^{m\times n}$ output matrix

$\boldsymbol{D} \in \Re^{m\times p}$ feedthrough matrix

- **Linear Homogeneous** (**LH**) state-space system

$$\Sigma_{\mathrm{H}} \quad \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(t)\boldsymbol{x}(t) & \textit{(state equation)} \\ \boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) & \textit{(output equation)} \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & \textit{(initial state)} \end{cases}$$

- This is as special case of LTV and LTI systems resulting after setting $\boldsymbol{u}(t) = \boldsymbol{0}$ for all values of t.

Linearization of Dynamic System

- **Objective**. Find a linear dynamic system that approximates a given nonlinear system

$$
\Sigma \quad \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) & (\textit{state equation}) \\ \boldsymbol{y}(t) = \boldsymbol{g}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) & (\textit{output equation}) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 & (\textit{initial state}) \end{cases}
$$

$\Downarrow$ ***Linearization about an operating point***

$$
\Sigma_\ell \quad \begin{cases} \dfrac{d\tilde{\boldsymbol{x}}}{dt} = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\tilde{\boldsymbol{u}}(t) & (\textit{state equation}) \\ \tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\tilde{\boldsymbol{u}}(t) & (\textit{output equation}) \\ \tilde{\boldsymbol{x}}(t_0) = \boldsymbol{0} & (\textit{ZERO initial state}) \end{cases}
$$

- **Step 1**. Identification of a steady-state operating point$(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})$:
    - Steady state condition:

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{f}(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})=\boldsymbol{0}$$

$$\bar{\boldsymbol{x}}=\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_n\end{bmatrix} \bar{\boldsymbol{u}}=\begin{bmatrix}\bar{u}_1\\ \bar{u}_2\\ \vdots\\ \bar{u}_p\end{bmatrix}$$

    - Infinite number of solutions $(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}})$
        - n equations and $n+p$ unknowns

$$\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}f_1(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n,\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_p)\\ f_2(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n,\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_p)\\ \vdots\\ f_n(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n,\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_p)\end{bmatrix}$$

- **Step 2**. Introduction of deviation variables

$$\tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \bar{\boldsymbol{x}}$$
$$\tilde{\boldsymbol{u}}(t) := \boldsymbol{u}(t) - \bar{\boldsymbol{u}}$$
$$\tilde{\boldsymbol{y}}(t) := \boldsymbol{y}(t) - \boldsymbol{g}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$$

  - *Remark*:

$$\frac{d\tilde{\boldsymbol{x}}}{dt} = \frac{d(\boldsymbol{x} - \bar{\boldsymbol{x}})}{dt} = \frac{d\boldsymbol{x}}{dt}$$

- **Step 3**. Expansion of the nonlinear state-equation in terms of a truncated Taylor series. **Solution:**

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \vdots \\ \dot{\tilde{x}}_n \end{bmatrix} = \begin{bmatrix} \overline{\frac{\partial f_1}{\partial x_1}} & \overline{\frac{\partial f_1}{\partial x_2}} & \cdots & \overline{\frac{\partial f_1}{\partial x_n}} \\ \overline{\frac{\partial f_2}{\partial x_1}} & \overline{\frac{\partial f_2}{\partial x_2}} & \cdots & \overline{\frac{\partial f_2}{\partial x_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\frac{\partial f_n}{\partial x_1}} & \overline{\frac{\partial f_n}{\partial x_2}} & \cdots & \overline{\frac{\partial f_n}{\partial x_n}} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} + \begin{bmatrix} \overline{\frac{\partial f_1}{\partial u_1}} & \overline{\frac{\partial f_1}{\partial u_2}} & \cdots & \overline{\frac{\partial f_1}{\partial u_p}} \\ \overline{\frac{\partial f_2}{\partial u_1}} & \overline{\frac{\partial f_2}{\partial u_2}} & \cdots & \overline{\frac{\partial f_2}{\partial u_p}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\frac{\partial f_n}{\partial u_1}} & \overline{\frac{\partial f_n}{\partial u_2}} & \cdots & \overline{\frac{\partial f_n}{\partial u_p}} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_p \end{bmatrix}$$

or

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\tilde{\boldsymbol{u}}(t)$$

- $$\overline{\frac{\partial f_i}{\partial x_j}} := \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_{j+1}, \cdots, \bar{x}_n \\ \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_j, \bar{u}_{j+1} \cdots, \bar{u}_p}} \qquad \overline{\frac{\partial f_i}{\partial u_k}} := \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, \bar{x}_{k+1}, \cdots, \bar{x}_n \\ \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_k, \bar{u}_{k+1}, \cdots, \bar{u}_p}}$$

- **Step 4**. Expansion of the nonlinear output-equation in terms of a truncated Taylor series. **Solution**:

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_m \end{bmatrix} = \begin{bmatrix} \overline{\dfrac{\partial g_1}{\partial x_1}} & \overline{\dfrac{\partial g_1}{\partial x_2}} & \cdots & \overline{\dfrac{\partial g_1}{\partial x_n}} \\ \overline{\dfrac{\partial g_2}{\partial x_1}} & \overline{\dfrac{\partial g_2}{\partial x_2}} & \cdots & \overline{\dfrac{\partial g_2}{\partial x_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\dfrac{\partial g_m}{\partial x_1}} & \overline{\dfrac{\partial g_m}{\partial x_2}} & \cdots & \overline{\dfrac{\partial g_m}{\partial x_n}} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} + \begin{bmatrix} \overline{\dfrac{\partial g_1}{\partial u_1}} & \overline{\dfrac{\partial g_1}{\partial u_2}} & \cdots & \overline{\dfrac{\partial g_1}{\partial u_p}} \\ \overline{\dfrac{\partial g_2}{\partial u_1}} & \overline{\dfrac{\partial g_2}{\partial u_2}} & \cdots & \overline{\dfrac{\partial g_2}{\partial u_p}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\dfrac{\partial g_m}{\partial u_1}} & \overline{\dfrac{\partial g_m}{\partial u_2}} & \cdots & \overline{\dfrac{\partial g_m}{\partial u_p}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$$

or

$$\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\tilde{\boldsymbol{u}}(t)$$

$$\overline{\frac{\partial g_i}{\partial x_j}} := \left.\frac{\partial g_i}{\partial x_j}\right|_{\substack{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_{j+1}, \cdots, \bar{x}_n \\ \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_j, \bar{u}_{j+1} \cdots, \bar{u}_p}} \qquad \overline{\frac{\partial g_i}{\partial u_k}} := \left.\frac{\partial g_i}{\partial u_k}\right|_{\substack{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, \bar{x}_{k+1}, \cdots, \bar{x}_n \\ \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_k, \bar{u}_{k+1} \cdots, \bar{u}_p}}$$

- **Example of linearization**

  - Linearize the dynamic system about a steady-state point

$$\begin{cases} \dot{x}_1 = -2_1 x(t)\, x_1(t) + 32\, u(t) \\ \dot{x}_2 = x_1(t) - x_2(t)^2 - 4\sqrt{u(t)} \end{cases}$$

about the steady-state point $\bar{\boldsymbol{x}}^T = \begin{bmatrix} 8 & 2 \end{bmatrix}$ and $\bar{u} = 1$.

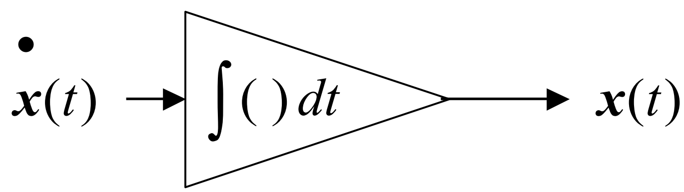
  - **Solution**: $\boldsymbol{A} = \begin{bmatrix} -4 & -16 \\ 1 & -4 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 32 \\ -2 \end{bmatrix} \boldsymbol{C} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \boldsymbol{D} = \begin{bmatrix} -18 \\ 0 \end{bmatrix}$

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - 8 \\ x_2 - 2 \end{bmatrix} \tilde{u} = u - 1 \qquad \tilde{\boldsymbol{y}} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 - 17 \\ y_2 - 8 \end{bmatrix}$$

Graphical Representation of LTI Systems

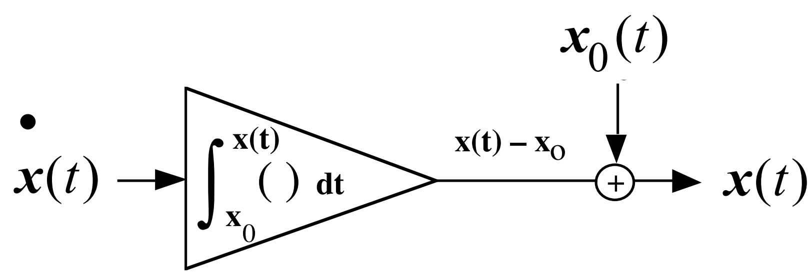
- **Integrator 1**: indefinite-integral operator

$$\int \dot{\boldsymbol{x}}(t)\,dt = \boldsymbol{x}(t)$$



- **Integrator 2**: definite-integral operator

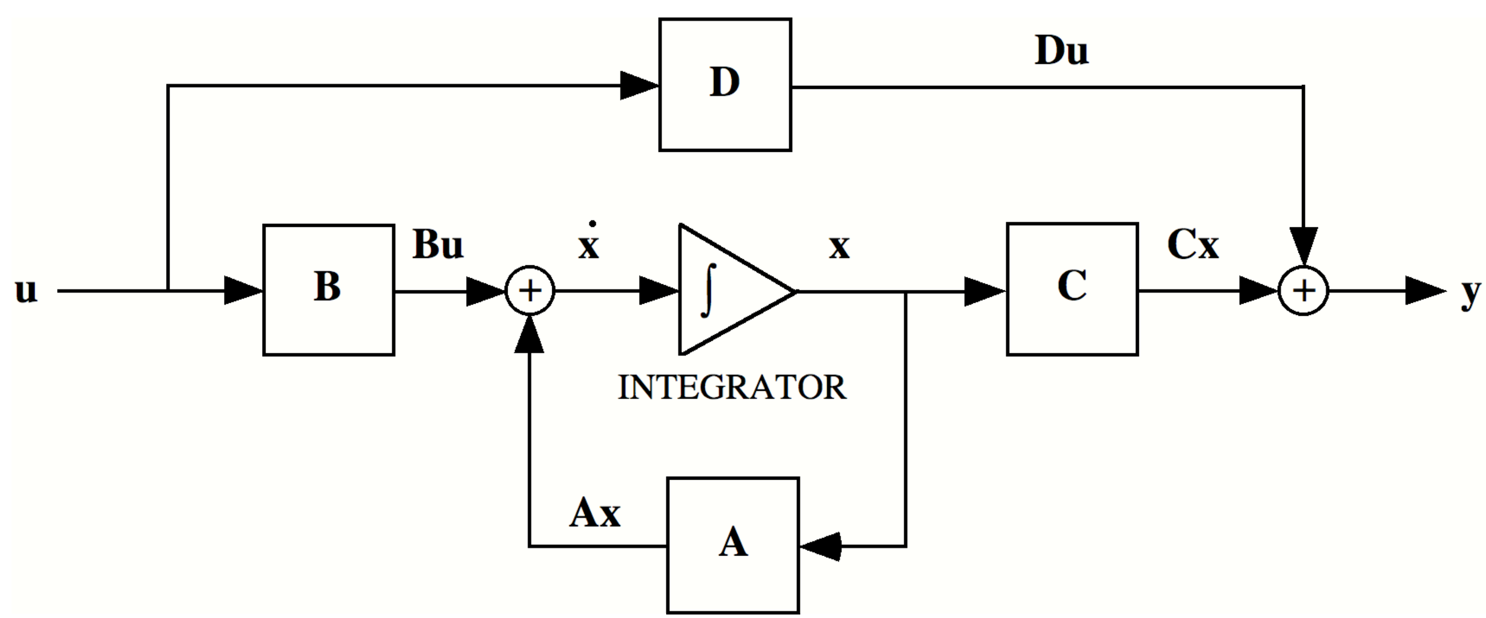
$$\int_{\boldsymbol{x}_0}^{\boldsymbol{x}(t)} \dot{\boldsymbol{x}}(t)\,dt = \boldsymbol{x}(t) - \boldsymbol{x}_0$$



- Mathematical representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$
$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

- Graphical representation

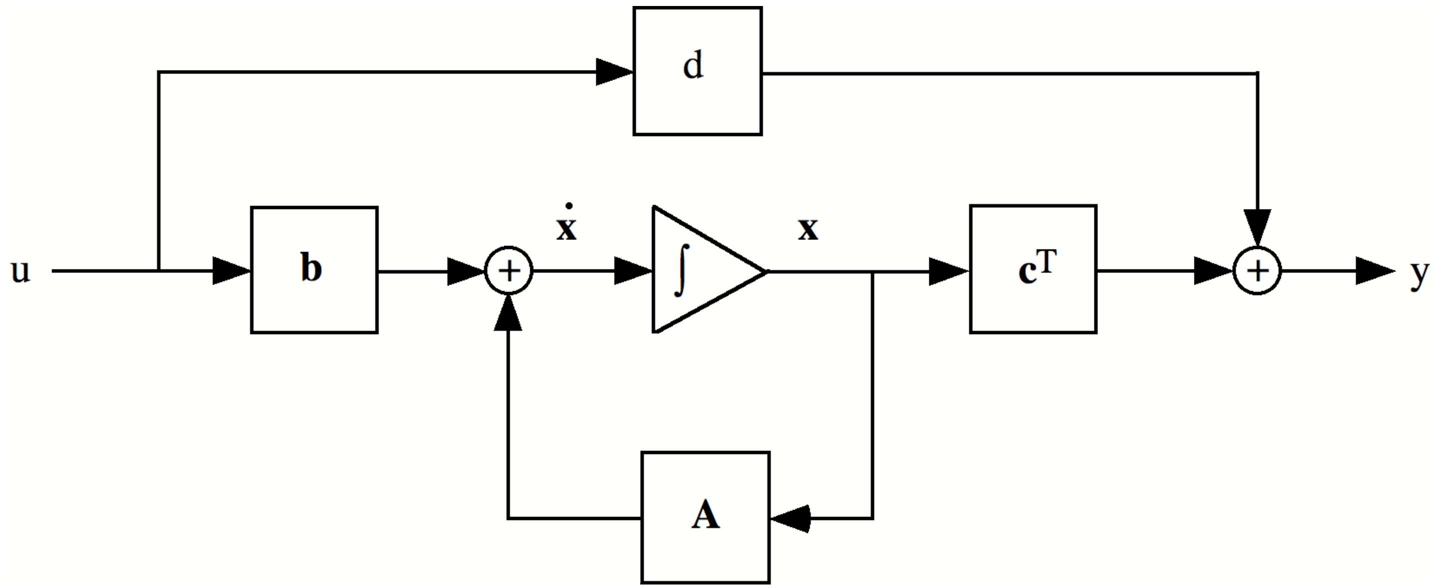


The SISO LTI System

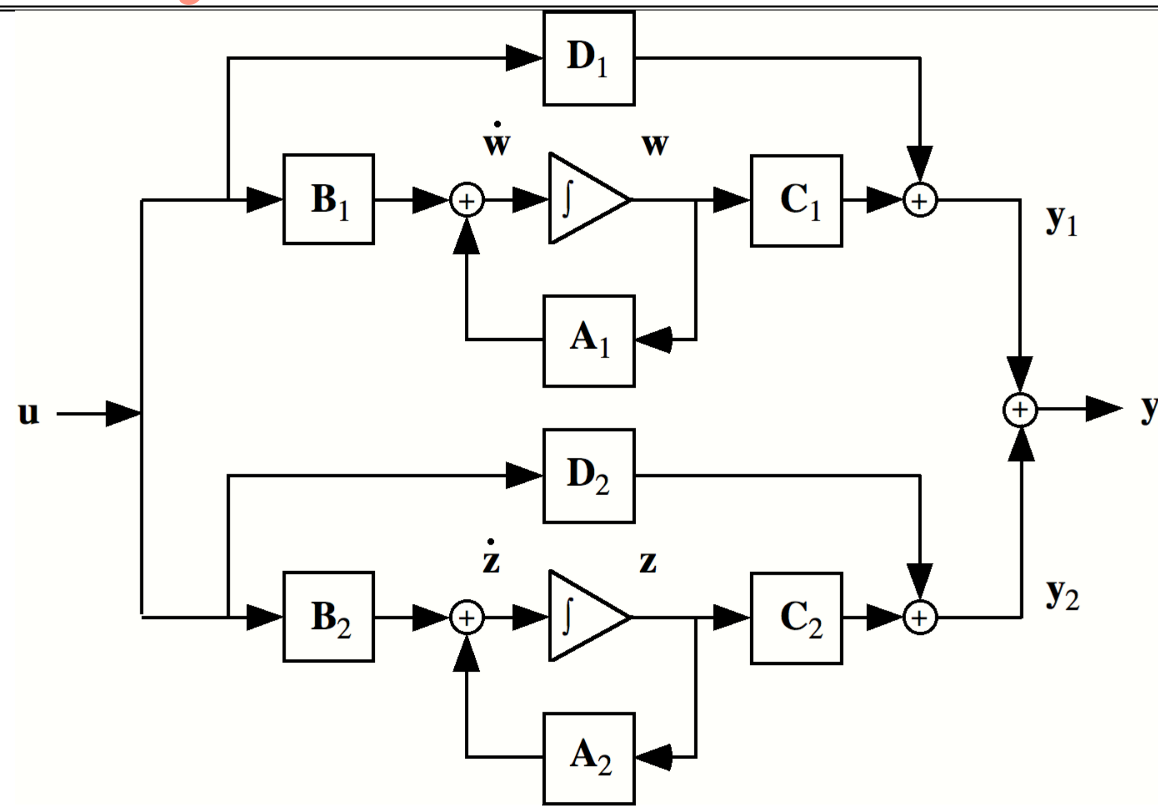
- Single-input/single-output: $u(t) \in \mathbb{R}\ (p = 1), \quad y(t) \in \mathbb{R}\ \ (m = 1)$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^{\mathrm{n}} \qquad \boldsymbol{c} := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^{\mathrm{n}} \qquad d \in \mathbb{R}$$

$$y(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(t) + du(t)$$



LTI Systems in Parallel



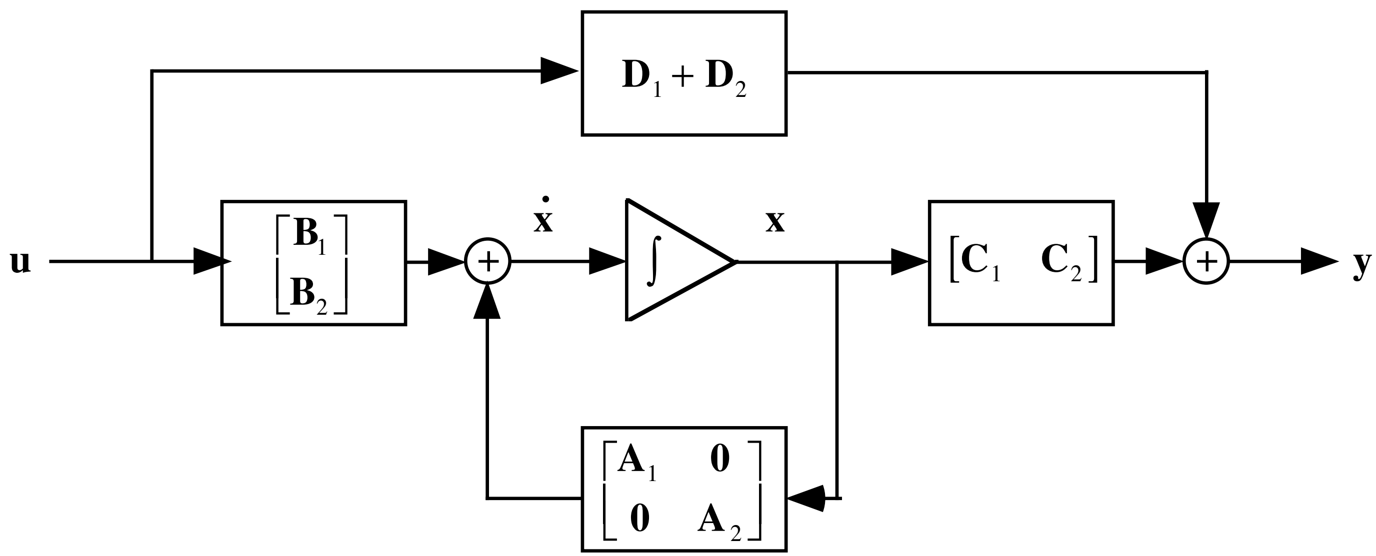
First System	Second System	Output Combination
$\dot{w} = A_1 w + B_1 u$	$\dot{z} = A_2 z + B_2 u$	$y = y_1 + y_2$
$y_1 = C_1 w + D_1 u$	$y = C_2 z + D_2 u$	

- Overall output equation

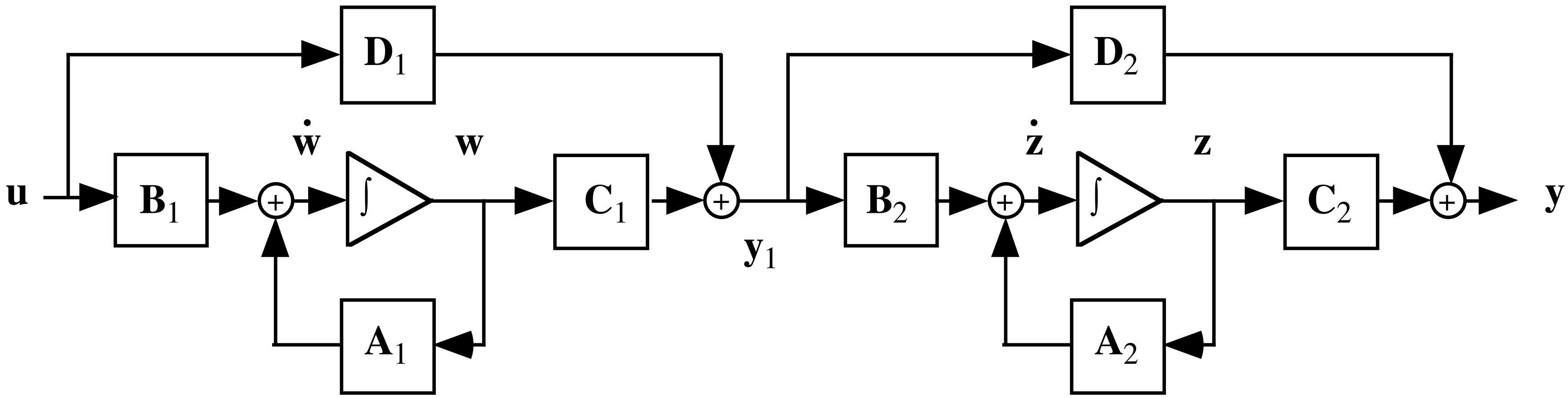
$$y = y_1 + y_2 = C_1 w + C_2 z + D_1 u + D_2 u = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} w \\ z \end{bmatrix} + \underbrace{\left(D_1 + D_2\right)}_{D} u$$

- Overall state equation

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A} \begin{bmatrix} w \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u$$



LTI Systems in Series



First System

$$\dot{w} = A_1 w + B_1 u$$

$$y_1 = C_1 w + D_1 u$$

Second System

$$\dot{z} = A_2 z + B_2 y_1$$

$$y = C_2 z + D_2 y_1$$

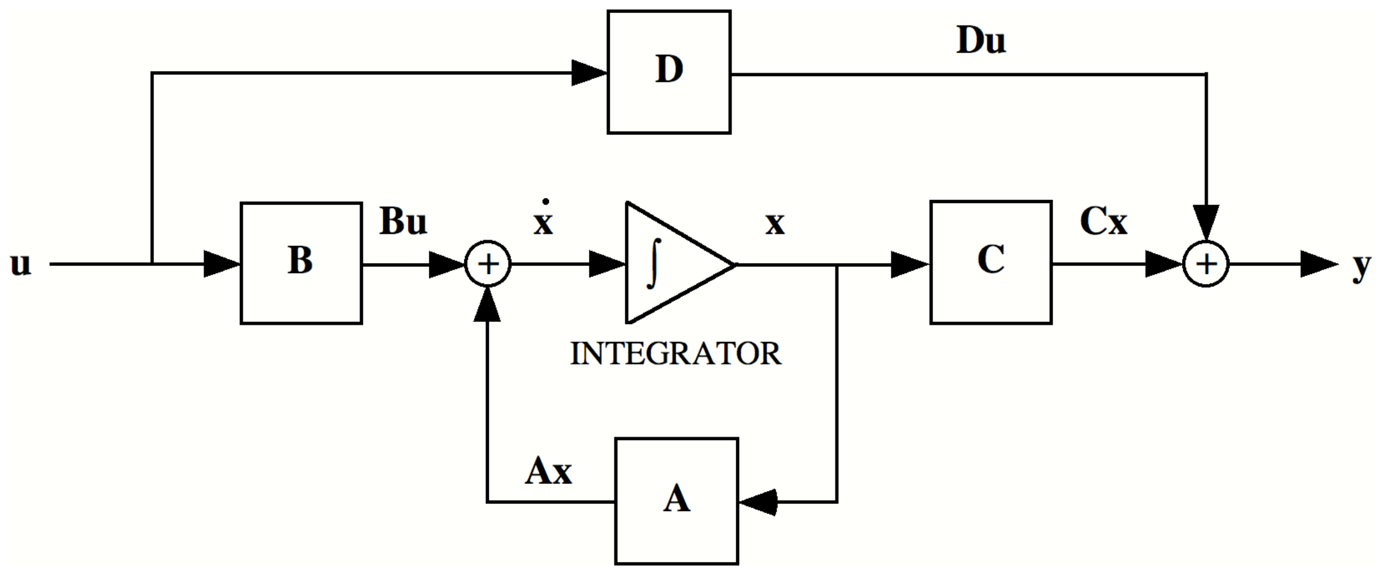
- Overall output equation

$$y = C_2 z + D_2 C_1 w + D_2 D_1 u = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} + (D_2 D_1) u = Cx + Du$$

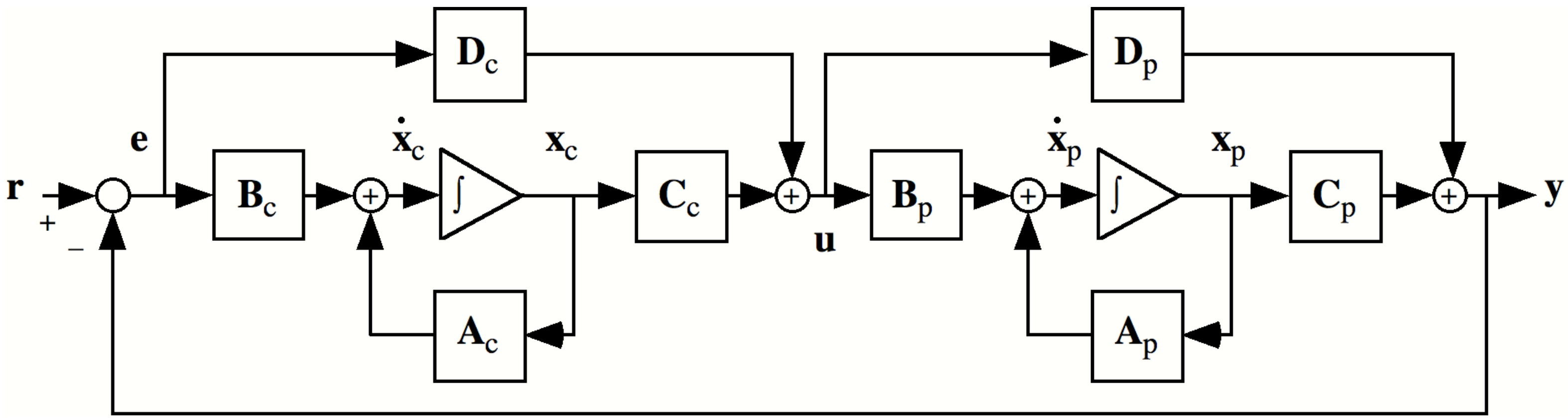
- Overall state equation

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$



LTI Systems in Negative Feedback Configuration



Process	Controller	Feedback Error
$\dot{\boldsymbol{x}}_p = \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p\boldsymbol{u}$	$\dot{\boldsymbol{x}}_c = \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c\boldsymbol{e}$	$\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{y}$
$\boldsymbol{y} = \boldsymbol{C}_p\boldsymbol{x}_p + \boldsymbol{D}_p\boldsymbol{u}$	$\boldsymbol{u} = \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\boldsymbol{e}$	

- **Well-posed system**: The inverse matrix $(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}$ exists
- Overall output equation

$$\boldsymbol{y} = \boldsymbol{C}_p\boldsymbol{x}_p + \boldsymbol{D}_p\left[\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c(\boldsymbol{r}-\boldsymbol{y})\right]$$
$$= \boldsymbol{C}_p\boldsymbol{x}_p + \boldsymbol{D}_p\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_p\boldsymbol{D}_c\boldsymbol{r} - \boldsymbol{D}_p\boldsymbol{D}_c\boldsymbol{y}$$

$$(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)\boldsymbol{y} = \boldsymbol{C}_p\boldsymbol{x}_p + \boldsymbol{D}_p\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_p\boldsymbol{D}_c\boldsymbol{r}$$

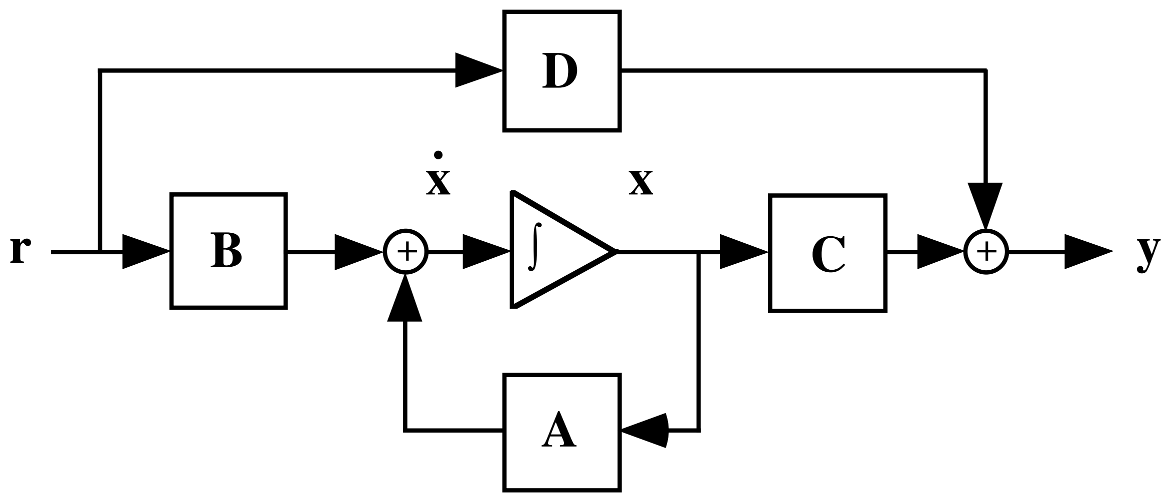
Given that the system is well-posed, it is possible to premultiply by $(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}$

$$\boldsymbol{y} = \begin{bmatrix}(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{C}_p & (\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{C}_c\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_p \\ \boldsymbol{x}_c\end{bmatrix} + (\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{D}_c\boldsymbol{r}$$
$$= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{r}$$

- Overall state equation

$$\begin{bmatrix} \dot{\boldsymbol{x}}_p \\ \dot{\boldsymbol{x}}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_p - \boldsymbol{B}_p\boldsymbol{D}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{C}_p & \boldsymbol{B}_p\boldsymbol{C}_c - \boldsymbol{B}_p\boldsymbol{D}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{C}_c \\ -\boldsymbol{B}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{C}_p & \boldsymbol{A}_c - \boldsymbol{B}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{C}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{x}_c \end{bmatrix}$$

$$+ \begin{bmatrix} \boldsymbol{B}_p\boldsymbol{D}_c - \boldsymbol{B}_p\boldsymbol{D}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{D}_c \\ \boldsymbol{B}_c - \boldsymbol{B}_c(\boldsymbol{I}+\boldsymbol{D}_p\boldsymbol{D}_c)^{-1}\boldsymbol{D}_p\boldsymbol{D}_c \end{bmatrix} \boldsymbol{r} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r}$$



(V) RESPONSE OF LTI SYSTEMS

by

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Contents

The State Transition Map

- Linear time-invariant state-space system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

$$\boldsymbol{x}(t_o) = \boldsymbol{x}_o$$

$$\boldsymbol{A} \in \mathbb{R}^{n\times n} \ (\textit{state matrix}), \qquad \boldsymbol{B} \in \mathbb{R}^{n\times p} \ (\textit{input matrix})$$

$$\boldsymbol{C} \in \mathbb{R}^{m\times n} \ (\textit{output matrix}), \qquad \boldsymbol{D} \in \mathbb{R}^{m\times p} \ (\textit{feedthrough matrix})$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \ (\textit{state}), \ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^p \ (\textit{input}), \ \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m \ (\textit{output})$$

- Solution to the differential equation for any input function $\mathbf{u}(t)$:

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_o)}\mathrm{x}_o + \int_{t_o}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\ \boldsymbol{u}(\tau)\ d\tau}$$

  - This relationship is known as the *variation-of-parameters formula* in the classical differential-equations literature
  - Known as the ***State Transition Map*** in the modern literature

    $\boldsymbol{u}(t)$ causes a transition from state $\boldsymbol{x}(t_o)$ to state $\boldsymbol{x}(t)$

- Solution to the overall LTI system

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_o)}\mathrm{x}_o + \int_{t_o}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\ \boldsymbol{u}(\tau)\ d\tau + \boldsymbol{D}\ \boldsymbol{u}(t)}$$

  - Known as the ***Output Map*** in the modern literature

Derivation of the VOP Formula

- Based on the strategy of using a integrating factor

- Given the state equation

$$\dot{\mathrm{x}}(t) = \mathrm{Ax}(t) + \mathrm{Bu}(t);\ \mathrm{x}(t_o) = \mathrm{x}_o$$

Pre-multiply both sides by $e^{-\mathrm{A}t}$ (*integrating factor*)

$$e^{-\mathrm{A}t}\frac{d\mathrm{x}}{dt} - e^{-\mathrm{A}t}\mathrm{Ax}(t) = e^{-\mathrm{A}t}\mathrm{Bu}(t)$$

Integrate both sides and use the property

$$\frac{d}{dt}(e^{-\mathrm{A}t}\mathrm{x}(t)) = e^{-\mathrm{A}t}\frac{d\mathrm{x}}{dt} - e^{-\mathrm{A}t}\mathrm{Ax}(t)$$

$$\int_{t_o, x_o}^{t, x(t)} d\left(e^{-A\tau} x(\tau)\right) = \int_{t_o}^{t} e^{-A\tau} \, Bu(\tau) d\tau$$

$$e^{-At} x(t) - e^{-At_o} x_o = \int_{t_o}^{t} e^{-A\tau} \, Bu(\tau) d\tau$$

$$e^{At} \left(e^{-At} x(t) - e^{-At_o} x_o \right) = e^{At} \left(\int_{t_o}^{t} e^{-A\tau} \, Bu(\tau) \, d\tau \right)$$

which yields

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_o)} x_o + \int_{t_o}^{t} e^{\boldsymbol{A}(t-\tau)} \boldsymbol{B} \, \boldsymbol{u}(\tau) \, d\tau}$$

- An expression for the matrix exponential function e^{At} is necessary to proceed

The State Transition Matrix

- The state transition matrix for an LTI system is defined as follows:

$$\Phi(t,t_0) := e^{\boldsymbol{A}(t-t_0)}$$

  - Properties
    - $\Phi(t,t) := \boldsymbol{I} \qquad [\ \Phi(t,t_0)\ ]^{-1} = \Phi(t_0,t)$

- State transition map and response map

$$\boxed{\begin{aligned} \boldsymbol{x}(t) &= \Phi(t,t_0)\ \mathrm{x}_o + \int_{t_o}^{t} \Phi(t,\tau)\ \boldsymbol{B}\ \boldsymbol{u}(\tau)\ d\tau \\ \boldsymbol{y}(t) &= \boldsymbol{C}\Phi(t,t_0)\ \mathrm{x}_o + \int_{t_o}^{t} \boldsymbol{C}\Phi(t,\tau)\ \boldsymbol{B}\ \boldsymbol{u}(\tau)\ d\tau + \boldsymbol{D}\ \boldsymbol{u}(t) \end{aligned}}$$

  - Expression also valid for LTV systems but **with a different formula for** $\Phi(t,t_0)$

Leibnitz's Rule

- Leibnitz's Rule for differentiation of integral functions

$$\frac{d}{dt}\int_{f(t)}^{g(t)} h(t,\tau)\,d\tau = \int_{f(t)}^{g(t)} \frac{\partial h(t,\tau)}{\partial t}\,d\tau + h\big(t, g(t)\big)\frac{dg(t)}{dt} - h\big(t, f(t)\big)\frac{df(t)}{dt}$$

- Application to the VOP formula:
  - Differentiation of the VOP expression should yield a derivative of the state that satisfies the state equation

- Derivative of the VOP formula

$$\frac{d}{dt}\boldsymbol{x}(t) = \frac{d}{dt}\left[e^{\boldsymbol{A}(t-t_o)}\mathrm{x}_o\right] + \frac{d}{dt}\left[\int_{t_o}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\ \boldsymbol{u}(\tau)\ d\tau\right]$$

Applying Leibnitz's rule to the second term

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}e^{\boldsymbol{A}(t-t_o)}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{A}e^{\boldsymbol{A}(t-\tau)}\ \boldsymbol{Bu}(\tau)d\tau + e^{\boldsymbol{A}(t-t)}\boldsymbol{Bu}(t)$$

$$= \boldsymbol{A}\left(e^{\boldsymbol{A}(t-t_o)}\boldsymbol{x}_o + \int_{t_o}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{Bu}(\tau)d\tau \right) + \boldsymbol{Bu}(t)$$

$$= \boldsymbol{Ax}(t) + \boldsymbol{Bu}(t)$$

hence, the VOP formula is a solution to the state equation

Similarity of LTI Systems

- Consider the LTI system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

and the "*change of basis*" transformation

$$\boldsymbol{x}(t) = \boldsymbol{H}\ \boldsymbol{z}(t) \qquad \Rightarrow \qquad \boldsymbol{z}(t) = \boldsymbol{H}^{-1}\ \boldsymbol{x}(t)$$

where $\boldsymbol{H} \in \mathbb{R}^{n\times n}$ is nonsingular

- Similar LTI system

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}\ \boldsymbol{z}(t) + \boldsymbol{H}^{-1}\boldsymbol{B}\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{H}\ \boldsymbol{z}(t) + \boldsymbol{D}\ \boldsymbol{u}(t) \end{aligned}$$

- Select $\boldsymbol{H}$ as a modal matrix, so that $\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \boldsymbol{M}$ is in canonical form
  - For example

$$\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} = diag(\boldsymbol{D}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{Q}) \coloneqq \boldsymbol{M}$$

so that

$$e^{\boldsymbol{A}t} = e^{\boldsymbol{H}\boldsymbol{M}\boldsymbol{H}^{-1}t} = \boldsymbol{H}e^{\boldsymbol{M}t}\boldsymbol{H}^{-1} = \boldsymbol{H}\begin{bmatrix} e^{\boldsymbol{D}t} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & e^{\boldsymbol{J}t} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & e^{\boldsymbol{K}t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & e^{\boldsymbol{Q}t} \end{bmatrix}\boldsymbol{H}^{-1}$$

$$= \boldsymbol{H}\, e^{diag(\boldsymbol{D}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{Q})\,t}\, \boldsymbol{H}^{-1}$$

where $e^{\boldsymbol{A}t} = \boldsymbol{H}\, e^{\boldsymbol{M}t}\boldsymbol{H}^{-1}$ is “easy” to calculate.

- VOP

$$\boldsymbol{x}(t) = \boldsymbol{H}\ e^{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}(t-t_o)}\boldsymbol{H}^{-1}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{H}\ e^{\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}(t-\tau)}\ \boldsymbol{H}^{-1}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

or

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{H}\ e^{\boldsymbol{M}(t-t_o)}\boldsymbol{H}^{-1}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{H}\ e^{\boldsymbol{M}(t-\tau)}\ \boldsymbol{H}^{-1}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}$$

- Output map

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{H}e^{\boldsymbol{M}(t-t_o)}\boldsymbol{H}^{-1}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{C}\ \boldsymbol{H}\ e^{\boldsymbol{M}(t-\tau)}\boldsymbol{H}^{-1}\boldsymbol{B}\ \boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\ \boldsymbol{u}(t)}$$

- **Summary**. Procedure for finding closed-form expressions for the state transition map and the response map for a given input $\boldsymbol{u}(t)$:
  - Find a modal matrix $\boldsymbol{H}$ for the given state matrix $\boldsymbol{A}$
  - Modify the state $\boldsymbol{A}$ matrix via the similarity transformation $\boldsymbol{M} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}$
    - The resulting similar matrix $\boldsymbol{M}$ is in canonical form
  - Find closed-form expressions for $e^{\boldsymbol{M}t}$ in terms of the eigenvalues, eigenvectors, and principal vectors of **A**
  - Substitute the closed-form expressions for $e^{\boldsymbol{M}t}$ and the known function $\boldsymbol{u}(t)$ into the expressions below and integrate:

$$\boldsymbol{x}(t) = \boldsymbol{H}\, e^{\boldsymbol{M}(t-t_o)}\boldsymbol{H}^{-1}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{H}\, e^{\boldsymbol{M}(t-\tau)}\boldsymbol{H}^{-1}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{H}e^{\boldsymbol{M}(t-t_o)}\boldsymbol{H}^{-1}\boldsymbol{x}_o + \int_{t_o}^{t} \boldsymbol{C}\,\boldsymbol{H}\, e^{\boldsymbol{M}(t-\tau)}\boldsymbol{H}^{-1}\boldsymbol{B}\,\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\,\boldsymbol{u}(t)$$

- Example of matrix exponential in canonical form

$$e^{\boldsymbol{M}\,t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & \frac{t^2}{2}e^{\lambda_4 t} & \frac{t^3}{3!}e^{\lambda_4 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & \frac{t^2}{2}e^{\lambda_4 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\lambda_4 t} & te^{\lambda_4 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\lambda_4 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}$$

Response Modes

$$x(t) = H\ e^{M(t-t_o)} H^{-1} x_o + \int_{t_o}^{t} H\ e^{M(t-\tau)} H^{-1} B u(\tau) d\tau$$

$$y(t) = CHe^{M(t-t_o)} H^{-1} x_o + \int_{t_o}^{t} C\ H\ e^{M(t-\tau)} H^{-1} B\ u(\tau) d\tau + D\ u(t)$$

- **Definition**. The response modes of an LTI system with state matrix A are the elements of the matrix exponential

$$e^{Mt} = \mathrm{e}^{diag(D, J, K, Q)t}$$

where

$$M = H^{-1} A\ H$$

- Modes introduced by ***D***
  - $e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}$
- Modes introduced by ***J***
  - $e^{\lambda_4 t}, te^{\lambda_4 t}, \frac{1}{2!}t^2 e^{\lambda_4 t}, \frac{1}{3!}t^3 e^{\lambda_4 t}$
- Modes introduced by ***K***
  - $e^{\sigma_5 t} \cos \omega_5 t,\ e^{\sigma_5 t} \sin \omega_5 t$
- Modes introduced by ***Q***
  - $e^{\sigma_6 t} \cos \omega_6 t,\ e^{\sigma_6 t} \sin \omega_6 t,\ t\, e^{\sigma_6 t} \cos \omega_6 t,\ t\, e^{\sigma_6 t} \sin \omega_6 t,$
  - $\frac{1}{2!}t^2 e^{\sigma_6 t} \cos \omega_6 t,\ \frac{1}{2!} e^{\sigma_6 t} \sin \omega_6 t$

- **Generalization**.
  - An eigenvalue

$$\lambda(A) = \mathrm{Re}\{\lambda(A)\} + j\, Im\{\lambda(A)\}$$

produces response modes of the forms

$$\boxed{\frac{1}{k!}\; t^k \;\; e^{\mathrm{Re}\{\lambda(A)\}\, t} \;\; cos\, Im\{\lambda(A)\}\; t}$$

and

$$\boxed{\frac{1}{k!}\; t^k \;\; e^{\mathrm{Re}\{\lambda(A)\}\, t} \;\; sin\, Im\{\lambda(A)\}\; t}$$

  - Remark
    - If $\mathrm{Re}\{\lambda(A)\} < 0$ then the response modes tend to zero
    - This is the key concept utilized to study the **stability** of LTI systems

(VI) THE MATRIX TRANSFER FUNCTION DESCRIPTION

by

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Contents

The Laplace Transform

- Preliminaries
  - $t \in \mathbb{R}$ — time variable
  - $s = \alpha + j\,\omega \in \mathbb{C}$ — complex variable, $\alpha, \omega \in \mathbb{R}$
- The complex exponential function

$$e^{-s\,t} = e^{-\alpha\,t}\cos(\omega t) - j\;e^{-\alpha\,t}\sin(\omega t)$$

- **Definition**. One-sided Laplace transform of a function of time

$$\boxed{\bar{\boldsymbol{f}}(s) := \int_0^\infty e^{-s\,t}\,\boldsymbol{f}(t)\;dt}$$

  - $\boldsymbol{f}(t)$ function of time — $\bar{\boldsymbol{f}}(s)$ Laplace transform of $\boldsymbol{f}(t)$
  - $(\,\boldsymbol{f}(t),\ \bar{\boldsymbol{f}}(s)\,)$ — Laplace-transform pair

- Notation $\bar{f}(s) = \mathcal{L}[f(t)]$
  - Simplified notation $f(s) = \mathcal{L}[f(t)]$

- Fundamental property of the Laplace Transform

  - $\mathcal{L}[\dot{f}(t)] = s\,\mathcal{L}[f(t)] - f(t)_{|t=0}$

    $\mathcal{L}[\dot{f}(t)] = s\,f(s) - f(t)_{|t=0}$

- Application to the solution of linear differential equations

  - Differential equation $\boxed{\dot{\boldsymbol{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$
  - Laplace Transform $s\,\mathbf{x}(s) - \mathbf{x}(t)_{|t=0} = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s)$
  - Algebraic equation $\boxed{(s\mathbf{I} - \mathbf{A})\mathbf{x}(s) = \mathbf{B}\mathbf{u}(s) + \mathbf{x}(t)_{|t=0}}$
  - Solution $\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{u}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}(t)_{|t=0}$

The differential equation is transformed into an algebraic equation

Transfer Function Representations

- Time-domain representation of an LTI dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{x}(t)_{|t=0} = \mathbf{x}_0$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)$$

- Zero initial conditions $\mathbf{x}(t)_{|t=0} = \mathbf{0}$
- Laplace transform of the dynamic system equations

$$(\mathbf{s\,I - A})\,\mathbf{x(s)} = \mathbf{B\,u(s) + 0}$$

$$\mathbf{y}(s) = \mathbf{C}\,\mathbf{x}(s) + \mathbf{D}\,\mathbf{u}(s)$$

  - Then $\mathbf{y}(s) = \left[\,\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\,\right]\mathbf{u}(s)$
  - **Definition**. Matrix transfer function **between the input and the output**

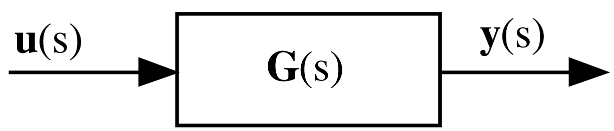
$$\mathbf{G}(s) := \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

- Laplace-domain representation of the LTI system

$$\boxed{\mathbf{y}(s) = \mathbf{G}(s)\,\mathbf{u}(s)}$$

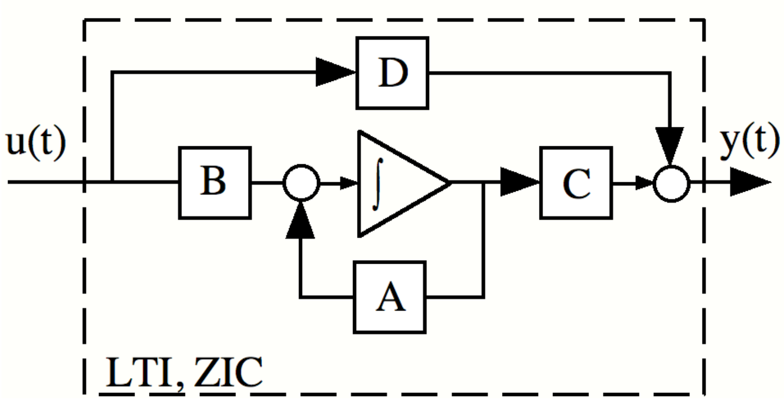
Input-output Relationship

$$\mathbf{y}(s) = \mathbf{G}(s)\,\mathbf{u}(s)$$



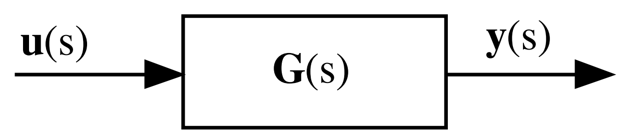
- Effect of the Laplace transformation

TIME-DOMAIN REPRESENTATION



$\Rightarrow$

INPUT-OUTPUT REPRESENTATION



Relevant Dimensions

- Size of time-domain signals $\boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{u} \in \mathbb{R}^p, \ \boldsymbol{y} \in \mathbb{R}^m$

- Size of system matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}, \ \boldsymbol{B} \in \mathbb{R}^{n \times p}, \ \boldsymbol{C} \in \mathbb{R}^{m \times n}, \ \boldsymbol{D} \in \mathbb{R}^{m \times p}$

- The matrix transfer function **G**(s) has ***m*** **x** ***p*** elements

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mp}(s) \end{bmatrix}$$

Structure of $G(s)$

- Definition $\mathbf{G}(s) := \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$
- Matrix inverse relationship $(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \dfrac{1}{det\,(s\boldsymbol{I} - \boldsymbol{A})}\,adj\,(s\boldsymbol{I} - \boldsymbol{A})$
- Detailed structure

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C}\,adj(s\boldsymbol{I}\text{-}\boldsymbol{A})\,\boldsymbol{B} + det(s\boldsymbol{I}\text{-}\boldsymbol{A})\,\boldsymbol{D}}{det(s\boldsymbol{I}\text{-}\boldsymbol{A})} = \frac{\boldsymbol{N}(s)}{det(s\boldsymbol{I}\text{-}\boldsymbol{A})}$$

  - $det(s\boldsymbol{I}\text{-}\boldsymbol{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$ polynomial (size 1x1)
  - $\boldsymbol{N}(s)$ matrix (size m x p)

$$\boldsymbol{N}(s) = \begin{bmatrix} n_{11}(s) & n_{12}(s) & \cdots & n_{1p}(s) \\ n_{21}(s) & n_{22}(s) & \cdots & n_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ n_{m1}(s) & n_{m2}(s) & \cdots & n_{mp}(s) \end{bmatrix}$$

- Final expression

$$\boldsymbol{G}(s) = \frac{1}{det(s\boldsymbol{I} - \boldsymbol{A})} \begin{bmatrix} n_{11}(s) & n_{12}(s) & \cdots & n_{1p}(s) \\ n_{21}(s) & n_{22}(s) & \cdots & n_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ n_{m1}(s) & n_{m2}(s) & \cdots & n_{mp}(s) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n_{11}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \frac{n_{12}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \cdots & \frac{n_{1p}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} \\ \frac{n_{21}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \frac{n_{22}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \cdots & \frac{n_{2p}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n_{m1}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \frac{n_{m2}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} & \cdots & \frac{n_{mp}(s)}{det(s\boldsymbol{I} - \boldsymbol{A})} \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mp}(s) \end{bmatrix}$$

Example

- Dynamic system

$$\dot{x}_1(t) = x_2(t) + 3\,u_1(t) + u_2(t)$$

$$\dot{x}_2(t) = -2x_1(t) - 3\,x_2(t) + u_1(t)$$

- Input-output representation

$$y_1(s) = \left(\frac{3s + 10}{s^2 + 3s + 2}\right)u_1(s) + \left(\frac{s^2 + 4s + 5}{s^2 + 3s + 2}\right)u_2(s)$$

$$y_2(s) = \left(\frac{s - 6}{s^2 + 3s + 2}\right)u_1(s) + \left(\frac{-2}{s^2 + 3s + 2}\right)u_2(s)$$

- Matrix transfer function

$$\boldsymbol{G}(s) = \begin{bmatrix} \dfrac{3s + 10}{s^2 + 3s + 2} & \dfrac{s^2 + 4s + 5}{s^2 + 3s + 2} \\ \dfrac{s - 6}{s^2 + 3s + 2} & \dfrac{-2}{s^2 + 3s + 2} \end{bmatrix}$$

The Resolvent

- The **resolvent** of a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is defined as $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$
- Standard expression

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{det\,(s\boldsymbol{I} - \boldsymbol{A})} = \frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{\lambda^{n} + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0}$$

- The resolvent formula (Useful for calculations and analytical studies)

$$(s\boldsymbol{I}\text{-}\boldsymbol{A})^{-1} = \frac{\sum_{j=0}^{n-1} s^{j} \sum_{i=j+1}^{n} \alpha_i\, \boldsymbol{A}^{i-j-1}}{det(s\boldsymbol{I}\text{-}\boldsymbol{A})}$$

- Inverse Laplace-transform of the resolvent
  - $\mathcal{L}\,[\,e^{\boldsymbol{A}t}\,] = (s\boldsymbol{I}\text{-}\boldsymbol{A})^{-1}$ $\qquad$ $\mathcal{L}^{-1}[\,(s\boldsymbol{I}\text{-}\boldsymbol{A})^{-1}\,] = e^{\boldsymbol{A}t}$
  - Laplace transform pair $\left(e^{\boldsymbol{A}t}\,,\,(s\boldsymbol{I}\text{-}\boldsymbol{A})^{-1}\right)$

Scalar Transfer Functions

- Single-input/single-output system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\ \boldsymbol{x}(t) + \boldsymbol{b}\ u(t) \qquad \boldsymbol{A} \in \mathbb{R}^{n\times n},\ \ \boldsymbol{b} \in \mathbb{R}^{n}$$

$$y(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(t) + d\ u(t) \qquad \boldsymbol{c} \in \mathbb{R}^{n},\ \ d \in \mathbb{R}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n} \qquad u \in \mathbb{R} \qquad y \in \mathbb{R}$$

- Scalar transfer function $y(s) = G(s)\ u(s)$

$$G(s) = \boldsymbol{c}^{\mathrm{T}}(\mathrm{s}\,\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d = \frac{n(s)}{det(s\boldsymbol{I} - \boldsymbol{A})}$$

Poles of a Transfer Function

- Scalar transfer function $G(s)$
  - **Definition**. A complex number $s = p$ is a pole of $G(s)$ if

$$\left| G(p) \right| \to \infty$$

- Matrix transfer function

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mp}(s) \end{bmatrix}$$

  - **Definition**. A complex number $s = p$ is a pole of $\boldsymbol{G}(s)$ if

$$\left| G_{ij}(p) \right| \to \infty$$

for at least one element (i, j) of $\boldsymbol{G}(s)$.

Examples of Poles

- Scalar transfer function
  - $$G(s) = \frac{3}{(s+1)(s+2)}$$ Poles: $p_1 = -1,\ p_2 = -2$
- Scalar transfer function
  - $$G(s) = \frac{3s+6}{s^2+3s+2} = \frac{3\ \cancel{(s+2)}}{(s+1)\ \cancel{(s+2)}}$$ Poles: $p_1 = -1$
- Matrix transfer function
  - $$\boldsymbol{G}(s) = \begin{bmatrix} \dfrac{3}{s} & \dfrac{3}{s+5} \\ \dfrac{3}{s-3} & \dfrac{3s+6}{s^2+3s+2} \end{bmatrix}$$ Poles: $p_1 = 0,\ p_2 = -5,\ p_3 = 3,\ p_4 = -1$

Minimality of Transfer Functions

- Scalar transfer function $G(s) = \dfrac{n(s)}{d(s)}$
  - **Definition**. The scalar transfer function G(s) is said to be minimal if the numerator and denominator polynomials have no common factors.
  - Examples:
    - $G(s) = \dfrac{3}{(s+1)(s+2)}$ MIMINAL
    - $G(s) = \dfrac{3s+6}{s^2+3s+2} = \dfrac{3\ \cancel{(s+2)}}{(s+1)\ \cancel{(s+2)}}$ NONMINIMAL

- Matrix transfer function

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mp}(s) \end{bmatrix}$$

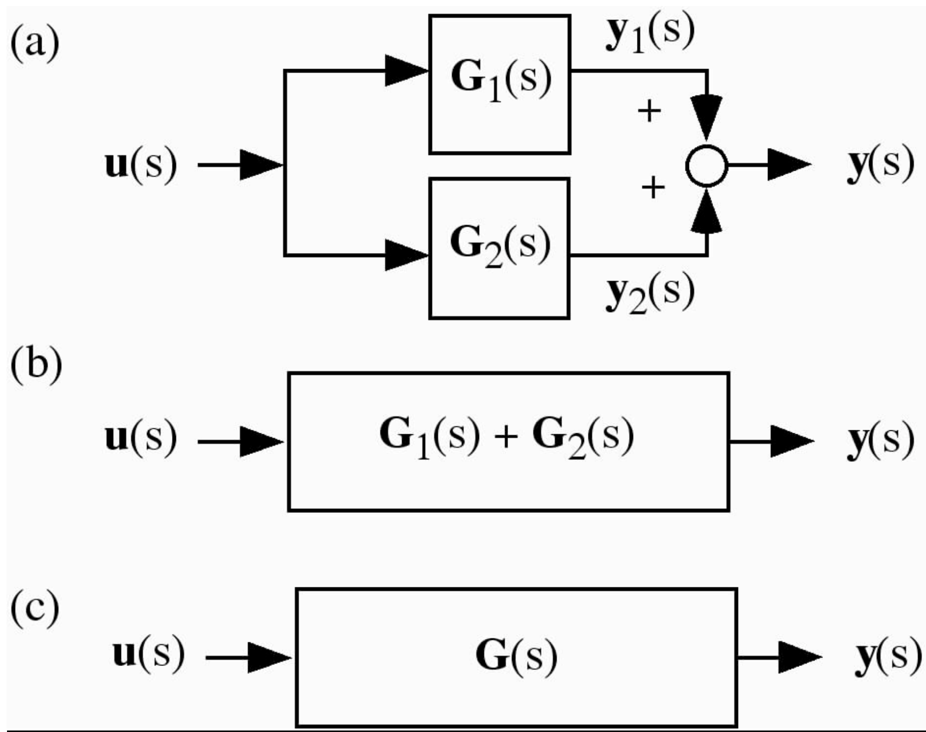
  - **Definition**. The matrix transfer function $\boldsymbol{G}(s)$ is said to be minimal if each of the *mp* SISO transfer functions $G_{ij}(s)$ is minimal, $i =$ 2, …, m, $j =$ 1, 2, …, p.

- Example

  - $$\boldsymbol{G}(s) = \begin{bmatrix} \dfrac{3}{s} & \dfrac{3}{s+5} \\ \dfrac{3}{s-3} & \dfrac{3s+6}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} \dfrac{3}{s} & \dfrac{3}{s+5} \\ \dfrac{3}{s-3} & \dfrac{3\ \cancel{(s+2)}}{(s+1)\ \cancel{(s+2)}} \end{bmatrix}$$ NON MINIMAL

  - $$\boldsymbol{G}(s) = \begin{bmatrix} \dfrac{3}{s} & \dfrac{3}{s+5} \\ \dfrac{3}{s-3} & \dfrac{3}{s+1} \end{bmatrix}$$ MINIMAL

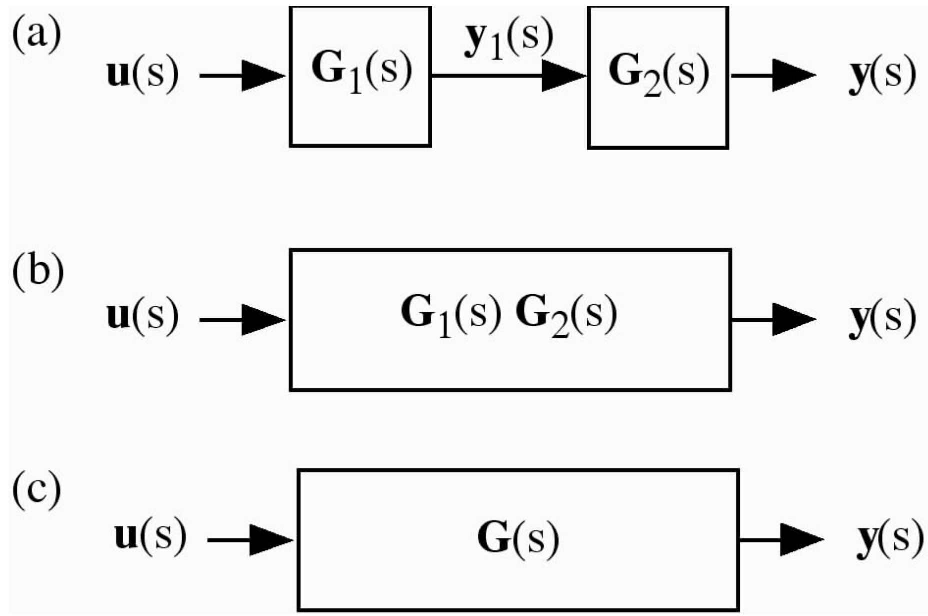
LTI Systems in Parallel



First System	Second System	Output Combination
$\boldsymbol{y}_1(s) = \boldsymbol{G}_1(s)\, \boldsymbol{u}(s)$	$\boldsymbol{y}(s) = \boldsymbol{G}_2(s)\, \boldsymbol{y}_1(s)$	$\boldsymbol{y}(s) = \boldsymbol{y}_1(s) + \boldsymbol{y}_2(s)$

$$\boxed{\boldsymbol{y}(s) = \left(\boldsymbol{G}_2(s) + \boldsymbol{G}_1(s) \right) \boldsymbol{u}(s)}$$

LTI Systems in Series



First System

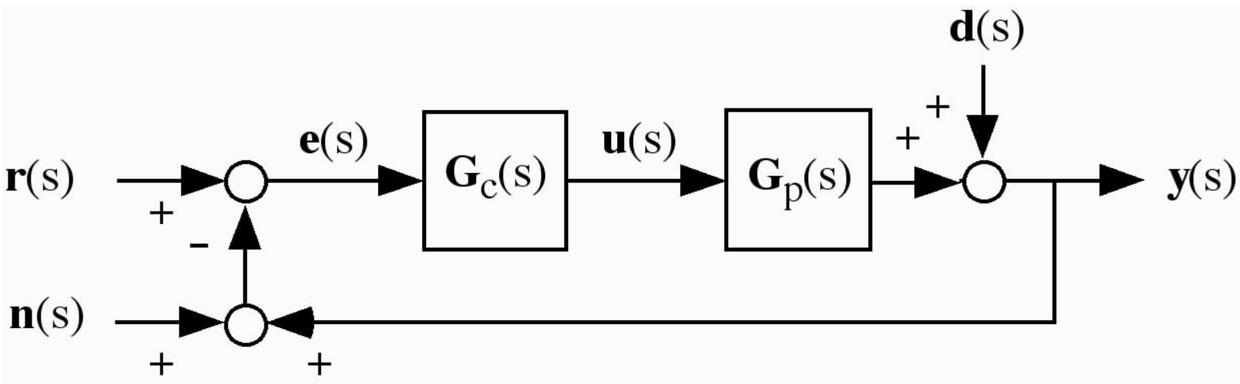
$$\boldsymbol{y}_1(s) = \boldsymbol{G}_1(s)\,\boldsymbol{u}(s)$$

Second System

$$\boldsymbol{y}(s) = \boldsymbol{G}_2(s)\,\boldsymbol{y}_1(s)$$

$$\boxed{\boldsymbol{y}(s) = \left(\,\boldsymbol{G}_1(s)\,\boldsymbol{G}_2(s)\,\right)\boldsymbol{u}(s)}$$

LTI Systems in Negative Feedback Configuration



$r(s)$	Set point
$d(s)$	Disturbance
$n(s)$	Measurement noise

Process	Controller	Feedback Error
$\boldsymbol{y}(s) = \boldsymbol{G}_p(s)\,\boldsymbol{u}(s)$	$\boldsymbol{u}(s) = \boldsymbol{G}_c(s)\,\boldsymbol{e}(s)$	$\boldsymbol{e}(s) = \boldsymbol{r}(s) - \boldsymbol{y}(s)$

- **Well-posed system**: The inverse matrix $(\,\boldsymbol{I} + \boldsymbol{G}_p(s)\,\boldsymbol{G}_c(s)\,)^{-1}$ exists as $s \to \infty$. *Remark*: $\boldsymbol{G}_p(\infty) = \boldsymbol{D}_p, \quad \boldsymbol{G}_c(\infty) = \boldsymbol{D}_c$

  - Well posed: There exists $(\,\boldsymbol{I} + \boldsymbol{G}_p(\infty)\,\boldsymbol{G}_c(\infty)\,)^{-1} = (\,\boldsymbol{I} + \boldsymbol{D}_p\;\boldsymbol{D}_c\,)^{-1}$

- Transfer functions between the set point and all the loop signals

$$\boxed{\begin{aligned}
\boldsymbol{y}(s) &= \left[\boldsymbol{I}+\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\right]^{-1}\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\ \boldsymbol{r}(s) &&= \boldsymbol{G}_{yr}(s)\ \boldsymbol{r}(s)\\
\boldsymbol{e}(s) &= \left[\boldsymbol{I}+\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\right]^{-1}\boldsymbol{r}(s) &&= \boldsymbol{G}_{er}(s)\ \boldsymbol{r}(s)\\
\boldsymbol{u}(s) &= \boldsymbol{G}_c(s)\left[\boldsymbol{I}+\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\right]^{-1}\boldsymbol{r}(s) &&= \boldsymbol{G}_{ur}(s)\ \boldsymbol{r}(s)
\end{aligned}}$$

- Closed-loop matrix transfer functions: $\boldsymbol{G}_{yr}(s)$, $\boldsymbol{G}_{er}(s)$, $\boldsymbol{G}_{ur}(s)$

- Classical definitions

  - Sensitivity function $\boxed{\boldsymbol{S}(s)=\left[\boldsymbol{I}+\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\right]^{-1}}$

  - Complementary Sensitivity: $\boxed{\boldsymbol{T}(s)=\left[\boldsymbol{I}+\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\right]^{-1}\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)}$

  - Closed-loop transfer functions

$$\boxed{\boldsymbol{y}(s)=\boldsymbol{T}(s)\ \boldsymbol{r}(s),\quad \boldsymbol{e}(s)=\boldsymbol{S}(s)\ \boldsymbol{r}(s),\quad \boldsymbol{u}(s)=\boldsymbol{G}_c(s)\boldsymbol{S}(s)\ \boldsymbol{r}(s)}$$

(VII) ANALYSIS: CONTROLLABILITY, OBSERVABILITY, AND STABILITY

by

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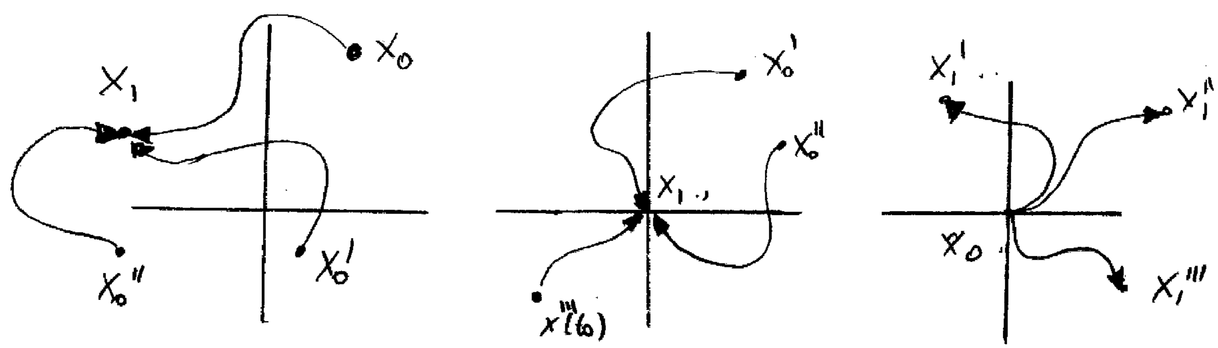
Contents

- Controllability and reachability
- Observability
- Stability
  - Stability in the sense of Lyapunov
  - External stability
  - Assessing stability

Controllability and Reachability

- Intuitive notions of controllability and reachability

  - Dynamic system
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$



General Controllability Concept

Controllability $\mathbf{x}(t_1) = \mathbf{0}$

Reachability $\mathbf{x}(t_0) = \mathbf{0}$

- **Controllability**. A dynamic state-space system is said to be controllable in the finite interval $[t_0, t_1]$ if for any initial state $\boldsymbol{x}(t_0)$ there exists an input $\boldsymbol{u}(t)$ such that $\boldsymbol{x}(t_1) = \boldsymbol{0}$

  - *Controllable*: $\big(\, t_0, \boldsymbol{x}(t_0) \,\big) \xrightarrow{\boldsymbol{u}(t),\ t \in [t_0,\, t_1]} \big(\, t_1, \boldsymbol{0} \,\big)$
  - *Notation*: The pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable

- **Reachability**. A dynamic state-space system is said to be reachable in the finite interval $[t_0, t_1]$ if for any final state $\boldsymbol{x}(t_1)$ there exists an input $\boldsymbol{u}(t)$ that transfers the initial state $\boldsymbol{x}(t_0) = \boldsymbol{0}$ to the final state $\boldsymbol{x}(t_1)$.

  - *Reachable*: $\big(\, t_0, \boldsymbol{0} \,\big) \xrightarrow{\boldsymbol{u}(t),\ t \in [t_0,\, t_1]} \big(\, t_1, \boldsymbol{x}(t_1) \,\big)$
  - *Notation*: The pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable

- **THEOREM. Kalman's controllability test.**

The LTI dynamic system with state matrix $\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ is controllable if and only if $rank(\boldsymbol{Q}_c) = n$, where

$$\boldsymbol{Q}_{\mathrm{c}} := \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \in \mathbb{R}^{n \times np}$$

- **THEOREM.**

The LTI dynamic system with state matrix $\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ is reachable if and only if it is controllable

$$\boxed{\text{Controllable LTI system} \Leftrightarrow \text{Observable LTI system}}$$

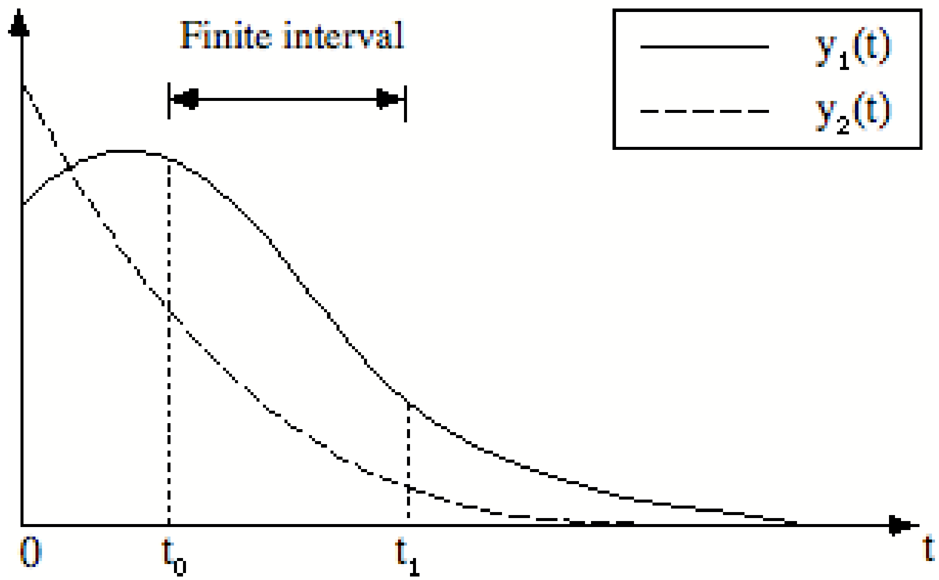
- Example

$$
R \quad \begin{cases} \dot{\boldsymbol{x}}(\mathrm{t}) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{u}(\mathrm{t}) \ , \ \boldsymbol{x}(0) = \boldsymbol{x}_{\mathrm{o}} = \begin{bmatrix} \boldsymbol{x}_{\mathrm{o}1} \\ \boldsymbol{x}_{\mathrm{o}2} \end{bmatrix} \\ \boldsymbol{y}(\mathrm{t}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) \end{cases}
$$

- Here n = 2 and $\boldsymbol{A}^{n-1}\boldsymbol{B} = \boldsymbol{AB} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $\boldsymbol{Q}_c = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad rank(\boldsymbol{Q}_c) = 1 = n$
- The LTI system (***A***, ***B***) is controllable.
- The LTI system (***A***, ***B***) is reachable.

Observability

- Intuitive notion of observability



$\boxed{\text{Find } \boldsymbol{x}(t_0)}$

- **Observability**. A dynamic state-space system is said to be observable in the finite interval $[t_0, t_1]$ if any initial state $\boldsymbol{x}(t_0)$ can be reconstructed from knowledge of the output function $\boldsymbol{y}(t)$ and the input function $\boldsymbol{u}(t)$ in the entire finite interval.

- *Observable*:

$$\left(\boldsymbol{y}(t), \boldsymbol{u}(t), \ t \in [t_0, t_1] \right) \xrightarrow{\text{reconstruction}} \boldsymbol{x}(t_0)$$

- *Notation*: The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable

● Observability for LTI systems

- It is conventional to set $\boldsymbol{u}(t) = \boldsymbol{0}, \ t \in [t_0, t_1]$

- *Observable*:

$$\left(\boldsymbol{y}(t), \boldsymbol{u}(t) = \boldsymbol{0}, \ t \in [t_0, t_1] \right) \xrightarrow{\text{reconstruction}} \boldsymbol{x}(t_0)$$

- *Notation*: The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable

- **THEOREM. Kalman's observability test.**

  The LTI dynamic system with state matrix $\boldsymbol{A}$ and output matrix $\boldsymbol{C}$ is observable if and only if

$$rank(\boldsymbol{Q}_o) = n$$

  where

$$\boldsymbol{Q}_\text{o} := \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA}^2 \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{bmatrix} \in \mathbb{R}^{nm \times n}$$

- *Remark*: $rank(\boldsymbol{Q}_o) = rank(\boldsymbol{Q}_o^\mathrm{T})$

$$\boldsymbol{Q}_o^\mathrm{T} := \begin{bmatrix} \boldsymbol{C}^\mathrm{T} & \boldsymbol{A}^\mathrm{T}\boldsymbol{C}^\mathrm{T} & (\boldsymbol{A}^2)^\mathrm{T}\boldsymbol{C}^\mathrm{T} & \cdots & (\boldsymbol{A}^{n-1})^\mathrm{T}\boldsymbol{C}^\mathrm{T} \end{bmatrix} \mathbb{R}^{n \times nm}$$

- Example

$$\begin{cases} \dot{\boldsymbol{x}}(\mathrm{t}) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{u}(\mathrm{t}) \ , \ \boldsymbol{x}(0) = \boldsymbol{x}_{\mathrm{o}} = \begin{bmatrix} \boldsymbol{x}_{\mathrm{o}1} \\ \boldsymbol{x}_{\mathrm{o}2} \end{bmatrix} \\ \boldsymbol{y}(\mathrm{t}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) \end{cases}$$

- $n = 2$ and $\boldsymbol{CA}^{\mathrm{n}-1} = \boldsymbol{CA} = \begin{bmatrix} 0\,1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$

- $\boldsymbol{Q}_{\mathrm{o}} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathrm{rank}\, \boldsymbol{Q}_{\mathrm{o}} = 2 = n$

- The LTI system ($\boldsymbol{A}$, $\boldsymbol{C}$) is observable

Stability of LTI Systems

- Two perspectives for analyzing the stability of the LTI system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

  - (1) **State stability** of the **ZERO INPUT** system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

    with respect to deviations from the *equilibrium point* $\boldsymbol{x}(0) = \boldsymbol{0}$

  - (3) **Bounded-input/bounded-output** (BIBO) stability of the **ZERO INITIAL STATE** system

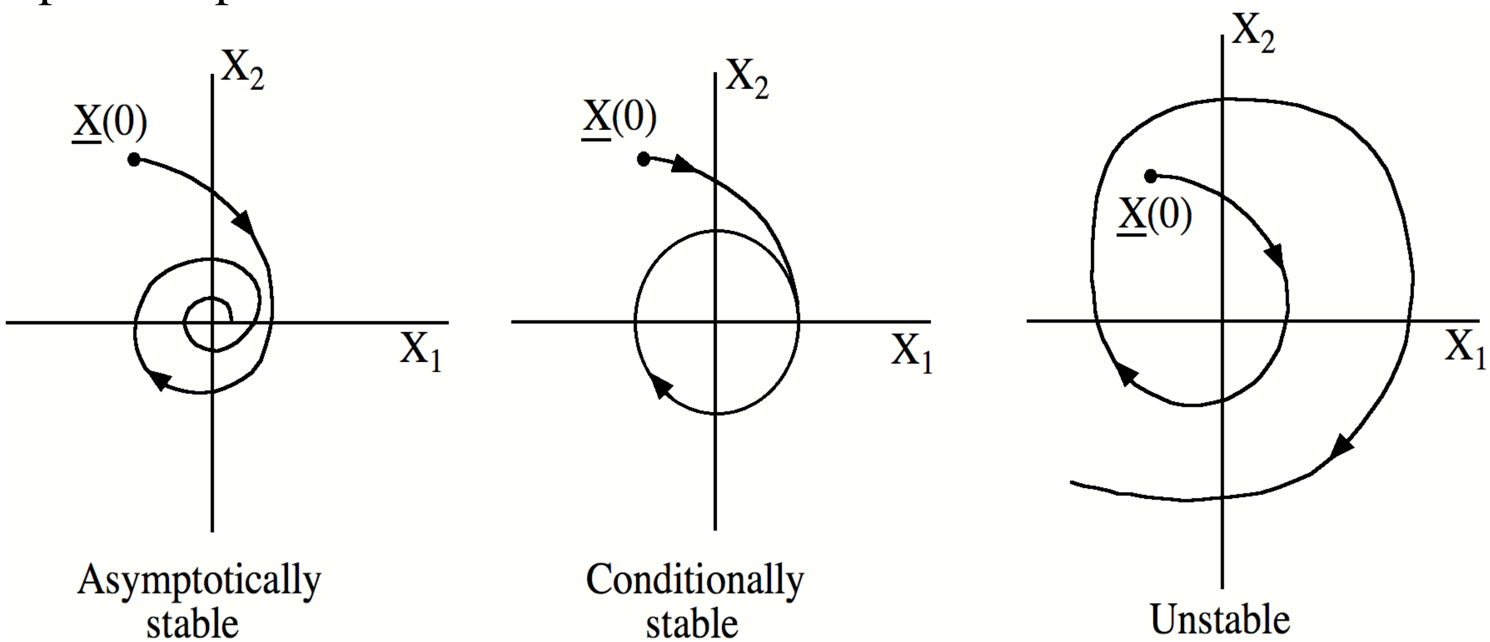
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{0}$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

State Stability

- State stability is concerned with the homogeneous (unforced) system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

- *Issue*: Is $\lim_{t\to\infty} \boldsymbol{x}(t)$ bounded for all initial conditions $\boldsymbol{x}_0 \neq \boldsymbol{0}$?
- Simple interpretation



- State trajectories (variation-of-parameters formula)

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

- Response modes (elements of $e^{\boldsymbol{A}t}$)
  - $\lambda = \sigma \in \mathbb{R} \quad \Rightarrow \quad e^{\lambda t},\ te^{\lambda t},\ \frac{1}{2!}t^2 e^{\lambda t},\ \frac{1}{3!}t^3 e^{\lambda t}, \cdots$
  - $\lambda = \sigma + j\omega \in \mathbb{C} \quad \Rightarrow \quad e^{\sigma t} cos\ \omega t,\ e^{\sigma t} sin\ \omega t,\ t\, e^{\sigma t} cos\ \omega t,\ t\, e^{\sigma t} sin\ \omega t, \cdots$
  - $\lambda = j\omega \in \mathbb{C} \quad \Rightarrow \quad cos\ \omega t,\ sin\ \omega t,\ t\ cos\ \omega t,\ t\ sin\ \omega t, \cdots$
    - $\omega = 0 \quad \Rightarrow \quad 1,\ t,\ t^2, \cdots$

- *Observation* # 1:
  - Eigenvalues with positive real parts ($\lambda_i(\boldsymbol{A}) \in \text{ORHP}$) produce response modes that are **unbounded** (they tend to infinity)

$$Re\big(\lambda_i(\boldsymbol{A})\big) > 0 \Rightarrow \{\, e^{\lambda_i t},\, te^{\lambda_i t},\, \frac{1}{2!}t^2 e^{\lambda_i t},\, \frac{1}{3!}t^3 e^{\lambda_i t},\, \cdots \} \rightarrow \{\, \infty, \infty, \infty, \infty, \cdots \}$$

- *Observation* # 2:
  - Eigenvalues with negative real parts ($\lambda_i(\boldsymbol{A}) \in \text{OLHP}$) produce response modes that are **bounded** (they tend to zero)

$$Re\big(\lambda_i(\boldsymbol{A})\big) < 0 \;\Rightarrow\; \{\, e^{\lambda_i t},\, te^{\lambda_i t},\, \frac{1}{2!}t^2 e^{\lambda_i t},\, \frac{1}{3!}t^3 e^{\lambda_i t},\, \cdots \} \rightarrow \{\, 0, 0, 0, 0, \cdots \}$$

- *Observation # 3*:
  - Eigenvalues with zero real parts ($\lambda_i(A) \in \mathrm{IA}$) produce response modes that are:
    - **Bounded** (values range between -1 and +1) if the geometric multiplicity is equal to the algebraic multiplicity

$$Re\big(\lambda_i(A)\big) = 0 \text{ and } g(\lambda_i) = a(\lambda_i)$$
$$\Rightarrow \{\, cos\ \omega t,\ sin\ \omega t \,\} \rightarrow \{\, [-1, +1], [-1, +1] \,\}$$

      - $\omega = 0 \text{ and } g(\lambda_i) = a(\lambda_i) \ \Rightarrow \ \{\, 1 \,\} \rightarrow \{\, 1 \,\}$
    - **Unbounded** (values tend to infinity) if the geometric multiplicity is less than the algebraic multiplicity

$$Re\big(\lambda_i(A)\big) = 0 \text{ and } g(\lambda_i) < a(\lambda_i) \Rightarrow \{\, t\ cos\ \omega t,\ t\ sin\ \omega t,$$
$$t^2 cos\ \omega t,\ t^2 sin\ \omega t, \cdots \,\} \rightarrow \{\, \infty, \infty, \infty, \infty, \cdots \,\}$$

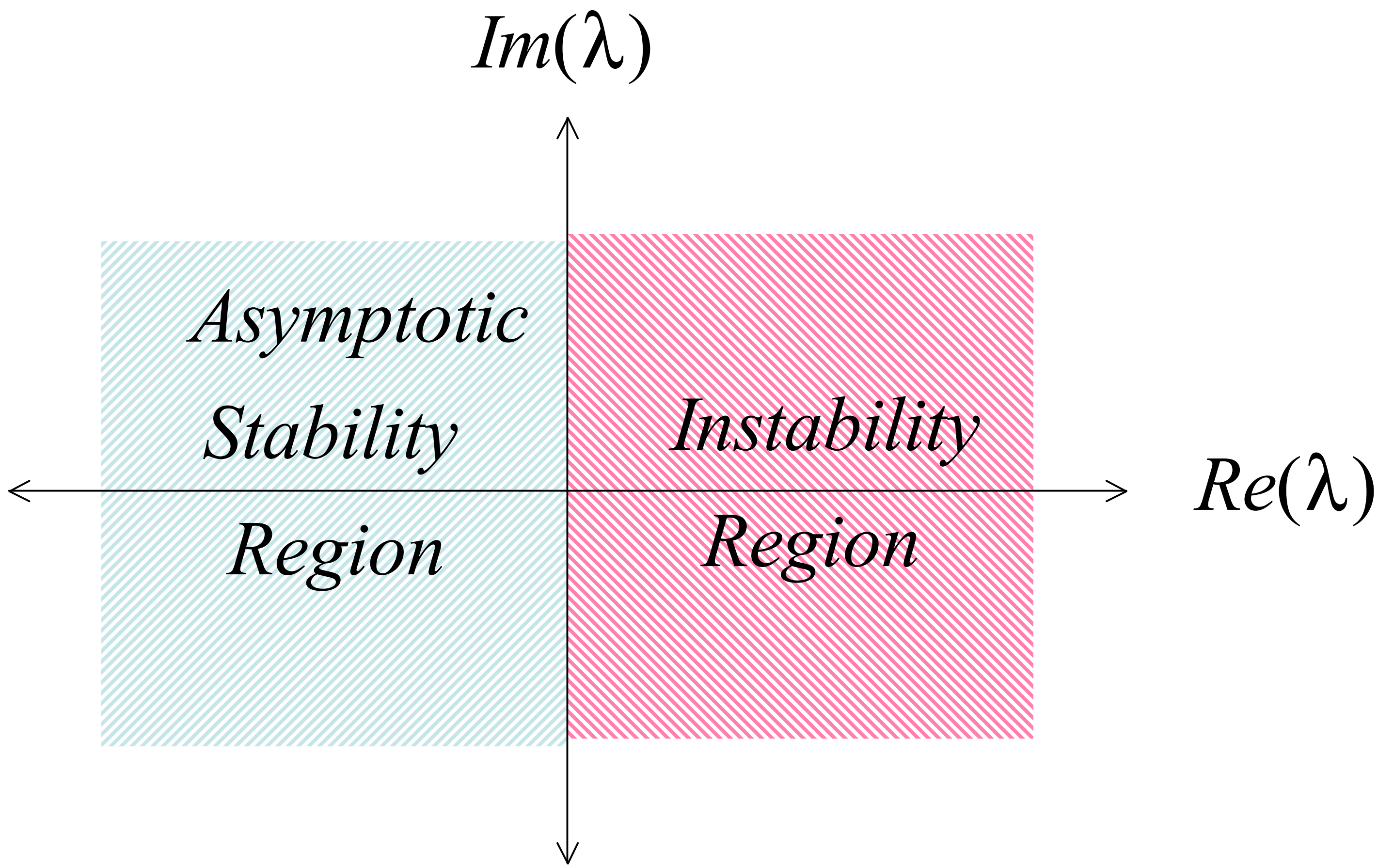
      - $\omega = 0 \text{ and } g(\lambda_i) < a(\lambda_i) \ \Rightarrow \ \{\, t, t^2, \cdots \,\} \rightarrow \{\, \infty, \infty, \cdots \,\}$

- **THEOREM. State Stability**

  - **<u>Asymptotic stability</u>**: all the eigenvalues of $\boldsymbol{A}$ have <u>negative</u> real parts.
    - *Asymptotic stability* $\Leftrightarrow$ $Re\big(\lambda_{\mathrm{i}}(\boldsymbol{A})\big) < 0 \quad i = 1,\, 2,\, \cdots,\, n$
  - **<u>Instability</u>**: One or more eigenvalues of $\boldsymbol{A}$ have <u>positive</u> real parts
    - *Instability* $\Leftrightarrow$ $\exists\, Re\big(\lambda_{\mathrm{i}}(\boldsymbol{A})\big) > 0$ for some $i = 1,\, 2,\, \cdots,\, n$
  - **<u>Conditional stability</u> (marginal stability, or stability in the sense of Lyapunov)**: All eigenvalues have either <u>negative or zero</u> real parts, and all eigenvalues that have a zero real part have geometric multiplicity equal to their algebraic multiplicity
    - *Conditional stability*

      $\Leftrightarrow$ $Re\big(\lambda_i(\boldsymbol{A})\big) \le 0$ for all $i = 1,\, 2,\, \cdots,\, n$

      and $g(\lambda_j) = a(\lambda_j)$ for j such that $Re\big(\lambda_j(\boldsymbol{A})\big) = 0$


Im(λ)
Asymptotic
Stability
Region
Instability
Region
Re(λ)

BIBO Stability

- BIBO stability is concerned with the forced LTI system with zero initial state (known as "zero state LTI system")

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{0}$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

- *Issue*: Given any bounded input function $\boldsymbol{u}(t)$

$$|u_1(t)| < \infty, \quad |u_2(t)| < \infty, \quad \cdots, \quad |u_p(t)| < \infty$$

will the output of the zero-state LTI system also be bounded?

$$||y_1(t)| < \infty, \quad |y_2(t)| < \infty, \quad \cdots, \quad |y_m(t)| < \infty \quad ?$$

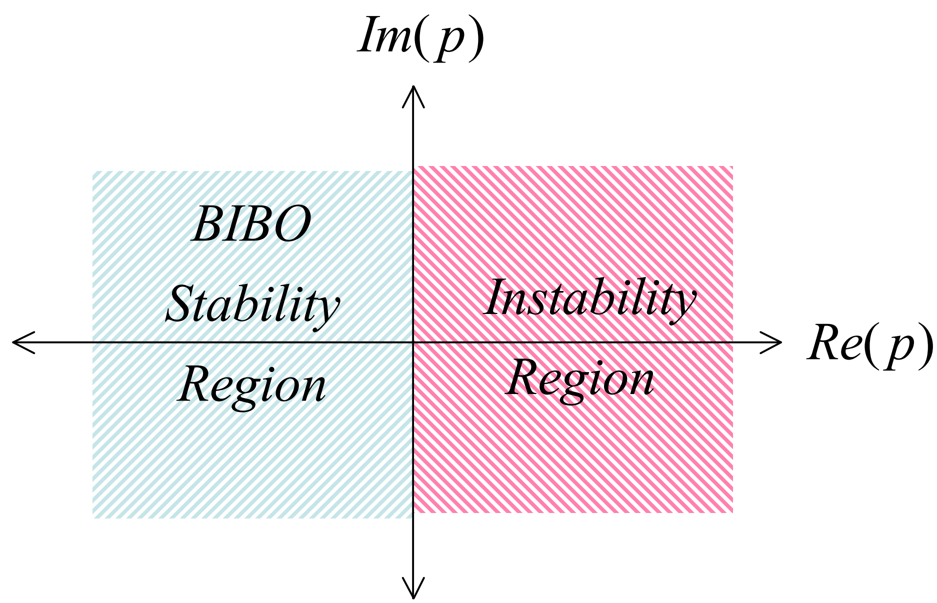
- Input-output representation

$$\boldsymbol{y}(s) = \boldsymbol{G}(s)\,\boldsymbol{u}(s) \quad \text{where} \quad \boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

- **Eigenvalue** $s = \lambda$ of the state matrix $\boldsymbol{A}$: $\boxed{det(s\boldsymbol{I} - \boldsymbol{A}) = 0}$
  - Total number of eigenvalues of $\boldsymbol{A}$: $\boxed{n}$
- **Poles** of matrix transfer function $\boldsymbol{G}(s) = \boldsymbol{C}\dfrac{adj(s\boldsymbol{I} - \boldsymbol{A})}{det(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{B} + \boldsymbol{D}$
  - **Pole** $s = p$ of $\boldsymbol{G}(s)$: $\boxed{\|\,\boldsymbol{G}(p)\,\| \to \infty}$
  - All eigenvalues of $\boldsymbol{A}$ are "candidate" poles because they make the denominator of $\boldsymbol{G}(s)$ equal to zero
  - Only the eigenvalues that do not cancel out with factors of the numerator are poles of $\boldsymbol{G}(s)$.
  - Total number of poles of $\boldsymbol{G}(s)$: $\boxed{n_p \leq n}$
- **Minimal transfer function:** $\boxed{n_p = n}$

- Example
  - $\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\boldsymbol{D} = 0$
  - *__Eigenvalues__*: $\Lambda(\boldsymbol{A}) = \{\ \lambda_1, \lambda_2\ \} == \{\ -1, -2\ \}$
  - $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 = \dfrac{1}{s+1}$
  - *__Poles__*: $\mathcal{P}(\ \boldsymbol{G}(s)\ ) = \{\ p_1\ \} == \{\ -1\ \}$
  - Eigenvalue $\lambda_1 = -1$ is a pole, but $\lambda_2 = -2$ is not a pole

- **THEOREM. BIBO Stability.**
  - **BIBO stability**: all the poles of $\boldsymbol{G}(\boldsymbol{s})$ have negative real parts.
    - *Asymptotic stability* $\quad \Leftrightarrow \quad Re\big(\, p_i(\boldsymbol{G}(s)) \,\big) < 0 \qquad i = 1,\ 2,\ \cdots,\ n_p$
  - **BIBO instability**: One or more poles of $\boldsymbol{G}(\boldsymbol{s})$ have positive real parts
    - *Instability* $\quad \Leftrightarrow \quad \exists\ Re\big(\, p_i(\boldsymbol{G}(s)) \,\big) > 0 \quad \text{for some } i = 1,\ 2,\ \cdots,\ n_p$



- Example
  - $\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\boldsymbol{D} = 0$
  - <u>*Eigenvalues*</u>: $\Lambda(\boldsymbol{A}) = \{\ \lambda_1, \lambda_2\ \} == \{\ -1, -2\ \}$
  - $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 = \dfrac{1}{s+1}$
  - <u>*Poles*</u>: $\mathcal{P}(\ \boldsymbol{G}(s)\ ) = \{\ p_1\ \} == \{\ -1\ \}$
  - Eigenvalue $\lambda_1 = -1$ is a pole, but $\lambda_2 = -2$ is not a pole
  - The LTI system is asymptotically stable because its pole has negative real part.

Assessing Stability

- Stability can be determined numerically by:
  - Calculation of eigenvalues (stability i.s.L.)
  - Calculation of poles (external stability)
  - ***Challenge***: the numerical computation of eigenvalues/poles can be numerically ill-conditioned for systems of large order.

- Methods that do not require the calculation of eigenvalues and poles
  - Routh-Array criteria
  - Lyapunov's first and second method for stability analysis

(VII) FEEDBACK CONTROL OF LTI SYSTEMS

by

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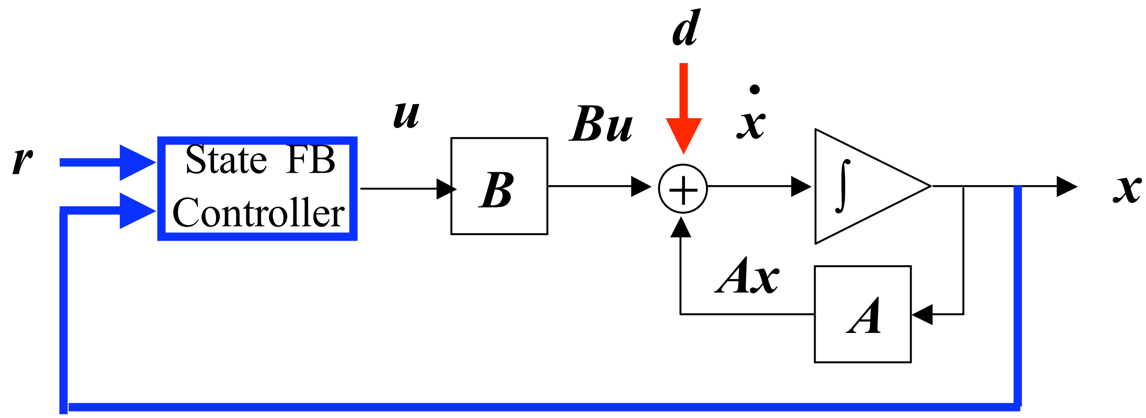
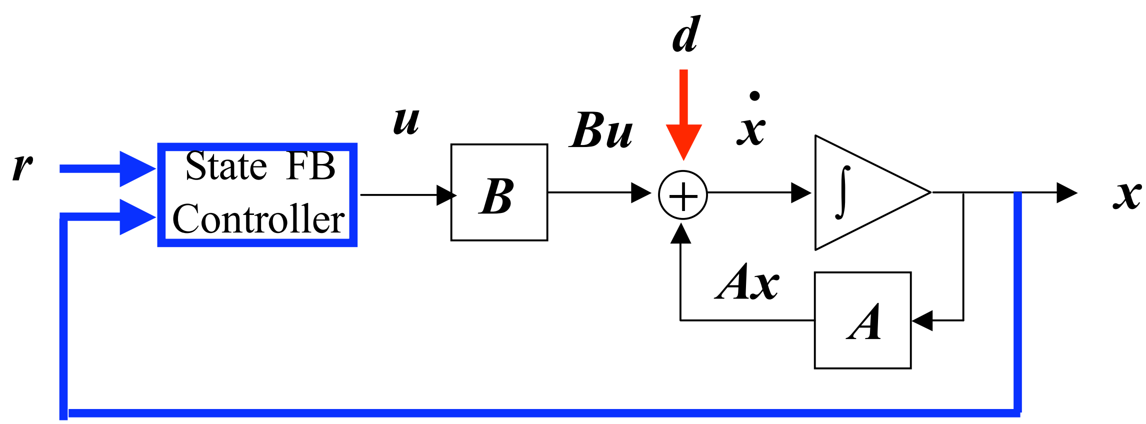
State Tracking and Regulation



- Open-loop LTI system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$
- Set-point signal (desired state values): $\boldsymbol{r}(t)$
  - Control objective $\boldsymbol{x}(t) \to \boldsymbol{r}(t)$
- **State Regulation** problem $\boldsymbol{r}(t) = \boldsymbol{0}$ for all $t \geq 0$
- **State Tracking** problem $\boldsymbol{r}(t) \neq \boldsymbol{0}$ for some $t \geq 0$

Disturbance Rejection



- External disturbance signal: $\boldsymbol{d}(t)$

- **Disturbance Rejection** Problem
  - Eliminate or minimize the effect of $\boldsymbol{d}(t)$ on the state tracking or regulation performance objective stated as $\boldsymbol{x}(t) \rightarrow \boldsymbol{r}(t)$

State Feedback Control Schemes

- LTI system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$

- State Feedback Control
  - Static Linear Feedback $\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t)$
  - Static Affine Feedback $\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t) + \boldsymbol{v}(t)$

    $\boldsymbol{v}(t)$ is dependent on $\boldsymbol{r}(t)$
  - Dynamic Feedback

$$\dot{\boldsymbol{x}}_{\mathrm{c}}(t) = \boldsymbol{A}_{\mathrm{c}}\boldsymbol{x}(t) + \boldsymbol{B}_{\mathrm{c}}\left(\ \boldsymbol{r}(t) - \boldsymbol{x}(t)\ \right)$$

$$\boldsymbol{u}(t) = \boldsymbol{C}_{\mathrm{c}}\boldsymbol{x}_{\mathrm{c}}(t) + \boldsymbol{D}_{\mathrm{c}}\left(\ \boldsymbol{r}(t) - \boldsymbol{x}(t)\ \right)$$

Regulation via State Feedback

- LTI system $\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)}$

- Linear State Feedback $\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t)$
  - Feedback gain $\boldsymbol{K} \in \mathbb{R}^{m\times n}$
  - Control objective $\boldsymbol{x}(t) \to \boldsymbol{0}$ (***Regulation***)

- Closed-loop system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\,(-\boldsymbol{K}\boldsymbol{x}(t)\,) = (\,\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}\,)\,\boldsymbol{x}(t)$

$$\boxed{\dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{A}}\;\boldsymbol{x}(t)}$$

  - Closed-loop state matrix $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$

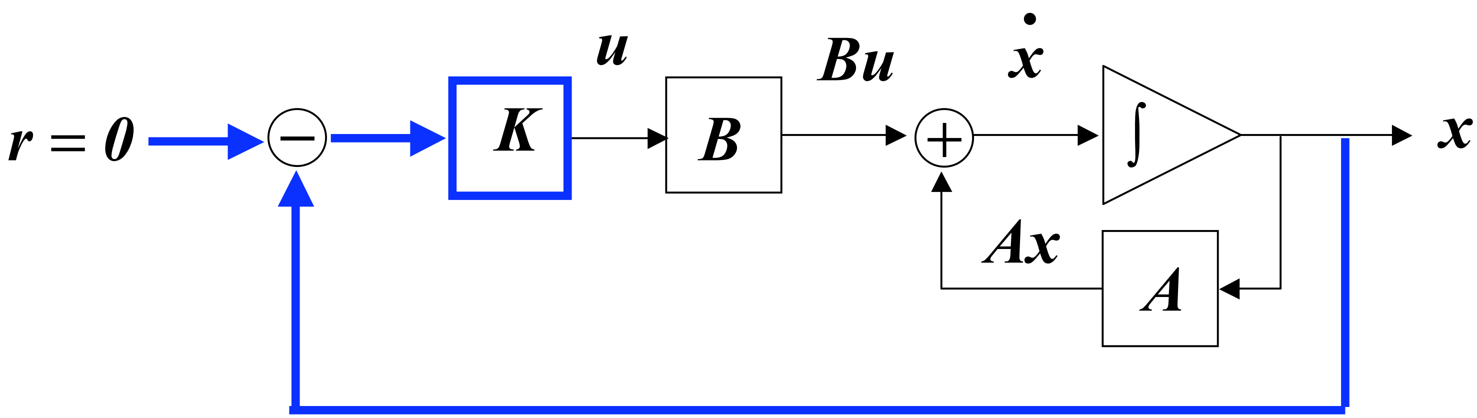
- Criteria for specification of the feedback gain $\boldsymbol{K}$
  - *Criterion No.* 1: $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{BK}$ is a stable matrix
  - *Criterion No.* 2: Meet desired time-domain specs, such as speed of decay to zero
- Common design methods
  - Trial-and-error design
    - Propose candidate $\boldsymbol{K}$
    - Determine if $\bar{\boldsymbol{A}}$ is stable
    - Assess time-domain performance via simulation
  - Eigenvalue-placement method
    - Find a gain $\boldsymbol{K}$ that places the eigenvalues of $\bar{\boldsymbol{A}}$ where desired
- Linear Quadratic Regulator (LQR) design
- Linear Quadratic Gaussian Regulator (LQG) design

State Regulation via Eigenvalue Placement

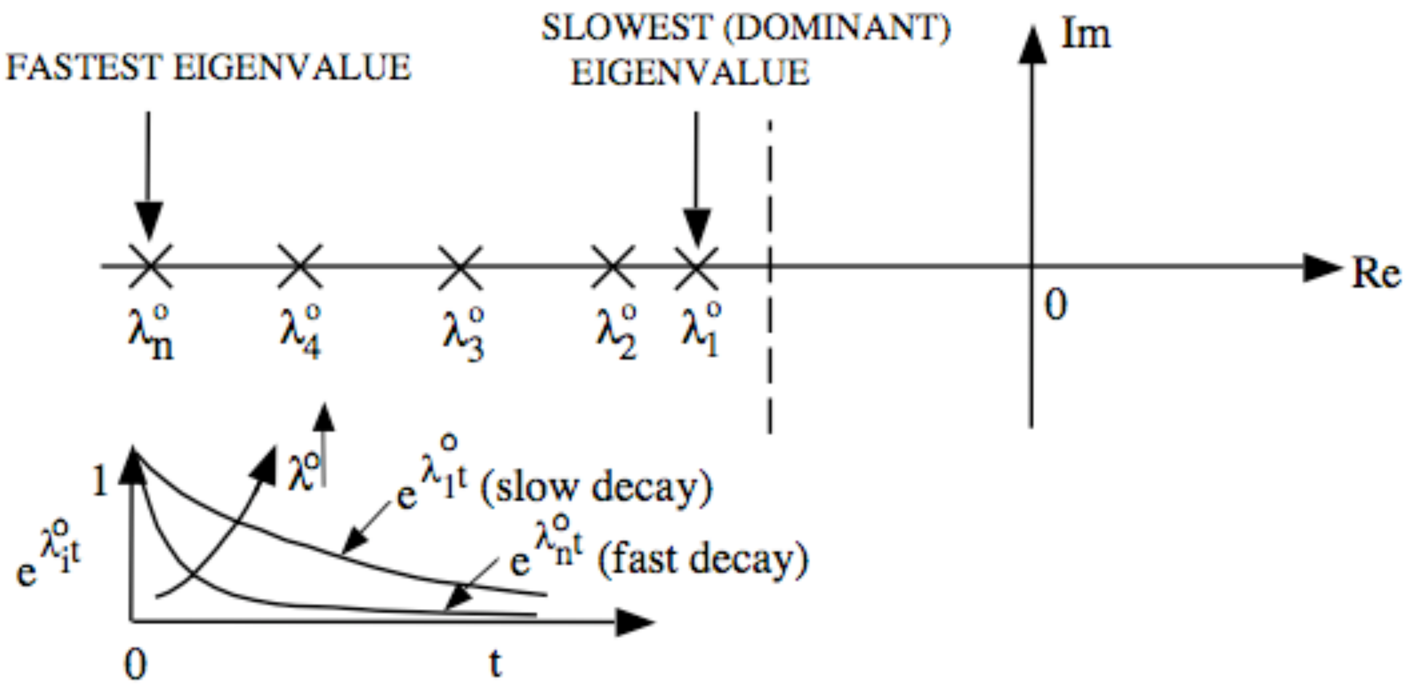
- LTI system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$
- Linear State Feedback $\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t)$
  - Feedback gain $\boldsymbol{K} \in \mathbb{R}^{m \times n}$
- Closed-loop state matrix $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$
- Eigenvalue-Placement Objective
  - Select $\boldsymbol{K}$ such that $\Lambda(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}) = \left\{\, \lambda_1^{\mathrm{o}}, \lambda_2^{\mathrm{o}}, \lambda_3^{\mathrm{o}}, \cdots, \lambda_n^{\mathrm{o}} \,\right\}$

$\left\{\, \lambda_1^{\mathrm{o}}, \lambda_2^{\mathrm{o}}, \lambda_3^{\mathrm{o}}, \cdots, \lambda_n^{\mathrm{o}} \,\right\}$ is a user-specified eigenvalue set

- Closed-loop Diagram



- Selection of closed-loop eigenvalues.
  - Place on the open left-half plane for stability
  - Speed of response: distance from the imaginary axis
  - Avoid complex poles: elimination of oscillatory behavior

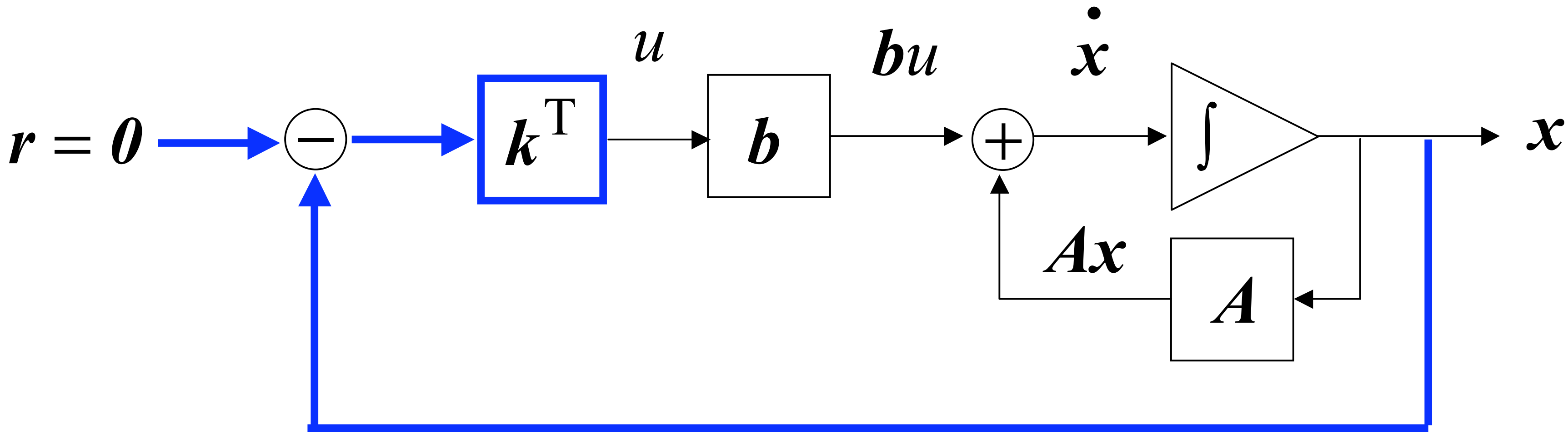


Pole Placement for Single-Input Systems

- Single-Input LTI system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$
- Dimensions $\boldsymbol{A} \in \mathbb{R}^{n\times n},\ \boldsymbol{b} \in \mathbb{R}^{n},\ \boldsymbol{x} \in \mathbb{R}^{n},\ u \in \mathbb{R}$
- Linear State Feedback $u(t) = -\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}(t)$
  - Feedback gain $\boldsymbol{k}^{\mathrm{T}} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \in \mathbb{R}^{1\times n}$
- Closed-loop state matrix $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}$
  - Remark

$$\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} = \begin{bmatrix} b_1k_1 & b_1k_2 & \cdots & b_1k_n \\ b_2k_1 & b_2k_2 & \vdots & b_2k_n \\ \vdots & \vdots & \ddots & \vdots \\ b_nk_1 & b_nk_2 & \cdots & b_nk_n \end{bmatrix}$$

- Closed-loop Diagram



- **EIGENVALUE-PLACEMENT THEOREM.**

  There exists a feedback matrix $\boldsymbol{k}^{\mathrm{T}} \in \mathbb{R}^{1\times n}$ that places the eigenvalues of $\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ at the arbitrary locations $\{\ \lambda_1^{\mathrm{o}},\ \lambda_2^{\mathrm{o}},\ \lambda_3^{\mathrm{o}},\ \cdots,\ \lambda_n^{\mathrm{o}}\ \}$ **if and only if** the pair ($\boldsymbol{A}$, $\boldsymbol{b}$) is **controllable**.

  - ***Remark 1***: Controllability $\Leftrightarrow rank(\boldsymbol{Q}_c) = n$

$$\boldsymbol{Q}_{\mathrm{c}} := \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

  - ***Remark 2***: The theorem applies only to sets $\{\ \lambda_1^{\mathrm{o}},\ \lambda_2^{\mathrm{o}},\ \lambda_3^{\mathrm{o}},\ \cdots,\ \lambda_n^{\mathrm{o}}\ \}$ where each complex eigenvalue appears as a complex conjugate pair

- Eigenvalue-Placement Methods
  - Trial and error
  - Reduction to canonical form
  - Ackermann's formula

Ackermann's Formula

- **Step 1**: Construct the controllability matrix

$$\boldsymbol{Q}_{\mathrm{c}} := \left[\boldsymbol{b} \;\; \boldsymbol{Ab} \;\; \boldsymbol{A}^2\boldsymbol{b} \;\; \cdots \;\; \boldsymbol{A}^{n-1}\boldsymbol{b} \right] \in \mathbb{R}^{n\times n}$$

- **Step 2**: Specify the set of desired closed-loop eigenvalues

$$\{\, \lambda_1^{\mathrm{o}},\, \lambda_2^{\mathrm{o}},\, \lambda_3^{\mathrm{o}},\, \cdots,\, \lambda_n^{\mathrm{o}} \,\}$$

- **Step 3**: Find the coefficients $\{\, \alpha_{\mathrm{o}}^{\mathrm{o}},\, \alpha_1^{\mathrm{o}},\, \alpha_2^{\mathrm{o}},\, \cdots,\, \alpha_{n\text{-}1}^{\mathrm{o}} \,\}$ of the desired characteristic polynomial

$$\begin{aligned} det(\boldsymbol{A} - \boldsymbol{bk}^{\mathrm{T}}) &= (\lambda - \lambda_1^{\mathrm{o}})\,(\lambda - \lambda_2^{\mathrm{o}})\,(\lambda - \lambda_3^{\mathrm{o}})\cdots(\lambda - \lambda_n^{\mathrm{o}}) \\ &= \lambda^n + \alpha_{n\text{-}1}^{\mathrm{o}}\lambda^{n\text{-}1} + \cdots + \alpha_3^{\mathrm{o}}\lambda^3 + \alpha_2^{\mathrm{o}}\lambda^2 + \alpha_1^{\mathrm{o}}\lambda + \alpha_{\mathrm{o}}^{\mathrm{o}} \end{aligned}$$

- **Step 4**: Define the polynomial matrix

$$\boldsymbol{P}(\boldsymbol{A}) = \boldsymbol{A}^n + \alpha^{\text{o}}_{n\text{-}1}\boldsymbol{A}^{n\text{-}1} + \cdots + \alpha^{\text{o}}_3\boldsymbol{A}^3 + \alpha^{\text{o}}_2\boldsymbol{A}^2 + \alpha^{\text{o}}_1\boldsymbol{A} + \alpha^{\text{o}}_{\text{o}}\boldsymbol{I} \in \mathbb{R}^{n\times n}$$

- **Step 5**: Find the feedback matrix using Ackermann's formula

$$\boldsymbol{k}^{\mathrm{T}} = \mathrm{FirstRow}(\, \boldsymbol{Q}_c^{-1}\ \boldsymbol{P}(\boldsymbol{A})\,)$$

or, equivalently

$$\boxed{\boldsymbol{k}^{\mathrm{T}} = [0 \quad 0 \quad \cdots \quad 1]\, \boldsymbol{Q}_c^{-1}\ \boldsymbol{P}(\boldsymbol{A})}$$

ACKERMANN'S FORMULA

FOR EIGENVALUE PLACEMENT

- *Remarks*
  - The concept of controllability plays a crucial role, since the inverse matrix $\boldsymbol{Q}_c^{-1} \in \mathbb{R}^{n\times n}$ exists because $rank(\boldsymbol{Q}_c) = n$
  - Extension to multiple-input systems ($p > 1$)
    - If the system is controllable, then one input can be used to place all the eigenvalues. The remaining inputs may be set to zero through the control law:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t) = -\begin{bmatrix} \boldsymbol{k}^{\mathrm{T}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x}(t)$$

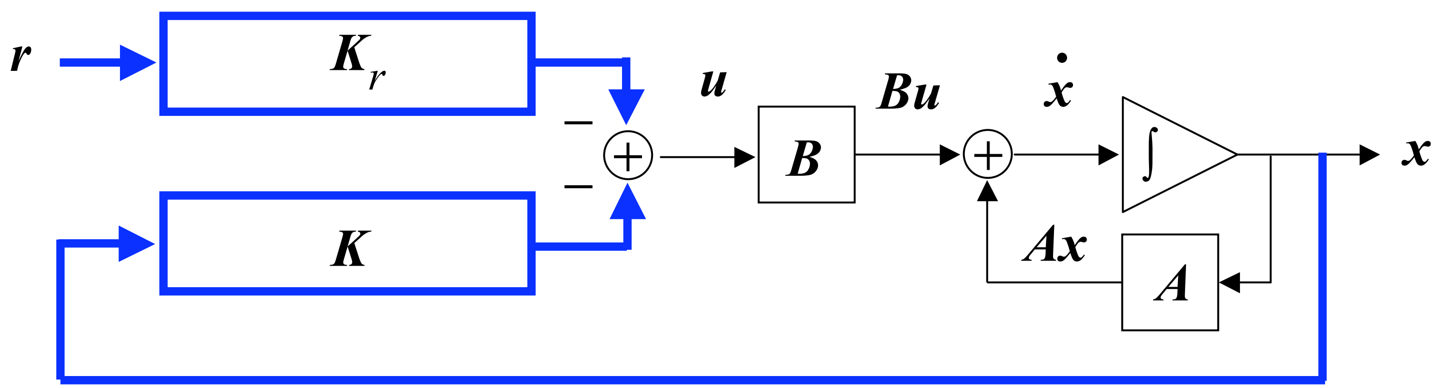
  - Numerical problems with Ackermann's formula
    - Ackermann's formula produces unreliable numerical results for large systems (usually for cases where $n > 10$) because it has a propensity for propagating and magnifying the small truncation errors made by digital computers.

Tracking via State Feedback

- LTI system $$\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)}$$
- State Feedback $\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t) - \boldsymbol{K}_r\,\boldsymbol{r}(t)$
  - **Feedback** gain $\boldsymbol{K} \in \mathbb{R}^{m\times n}$
  - **Feedforward** gain $\boldsymbol{K}_r \in \mathbb{R}^{m\times n}$
  - Control objective $\boldsymbol{x}(t) \to \boldsymbol{r}(t)$ (***Tracking***)
- Closed-loop system $\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{K}_r\boldsymbol{r}(t)$

$$\boxed{\dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{A}}\,\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{K}_r\boldsymbol{r}(t)}$$

  - Closed-loop state matrix $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$



- Design Criteria
  - Specification of the feedback gain $\boldsymbol{K}$ **(use regulation results)**
    - *Criterion No*. 1: $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{BK}$ is a stable matrix
    - *Criterion No*. 2: Adjust speed of response
  - Specification of the feedforward gain $\boldsymbol{K}_r$
    - *Criterion No*. 3: Ensure zero steady-state offset

$$\boxed{\text{steady}-\text{state offset} := \lim_{t \to \infty} [\, \boldsymbol{r}(t) - \boldsymbol{x}(t) \,]}$$

- Class of admissible set-points $\boldsymbol{r}(t) \in \mathbb{R}^n$
  - Bounded (does not become infinitely large)
  - Constant final value: $\lim_{t \to \infty} \boldsymbol{r}(t) = \boldsymbol{r}_s$

- **Necessary conditions** for existence of zero-offset for arbitrary admissible set points
  - (i) The state matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is nonsingular
  - (ii) The input matrix $\boldsymbol{B} \in \mathbb{R}^{n \times p}$ is full row-rank ($i.e.$ $p \geq n$)

- Implications of the necessary conditions
  - (i) $\Rightarrow$ there exists $\boldsymbol{A}^{-1}$
  - (ii) $\Rightarrow$ there exists $\boldsymbol{B}^{-\mathrm{R}} :\equiv \begin{cases} \boldsymbol{B}^{-1} & \text{if } p = n \\ \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}})^{-1} & \text{if } p > n \end{cases}$

- **OFSET-FREE TRACKING THEOREM**

  - **If** the necessary conditions (*i*) and (*ii*) are satisfied, and the feedback gain matrix $\boldsymbol{K}$ is specified such that the eigenvalues of $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ lie strictly in the open left-half plane.

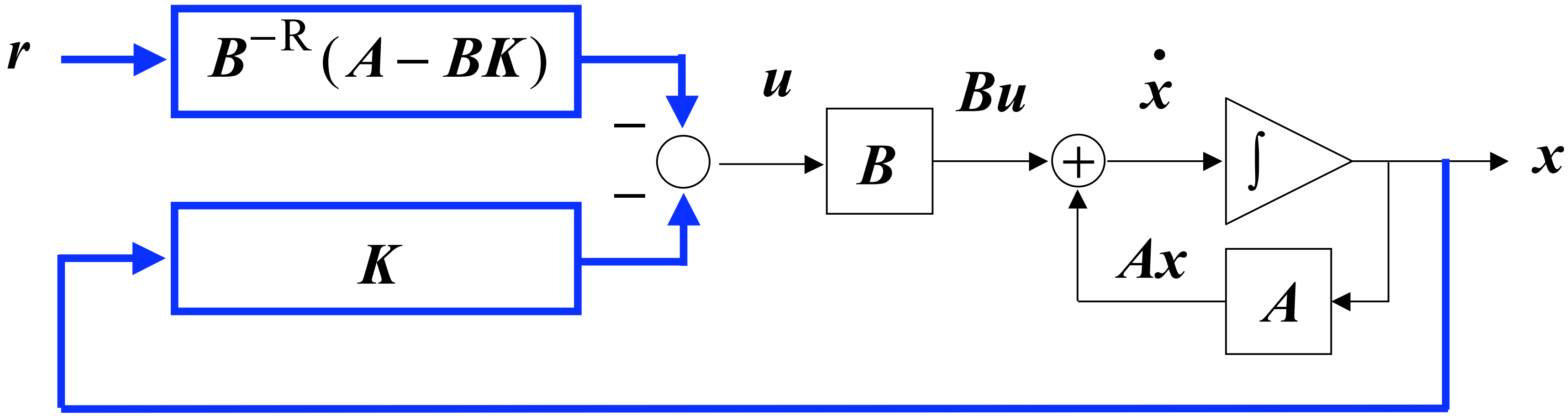
  - **Then** the state-feedback control law

$$\boldsymbol{u}(t) = -\boldsymbol{K}\,\boldsymbol{x}(t) - \boldsymbol{B}^{-\mathrm{R}}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{r}(t)$$

ensures that for every admissible set point

$$\lim_{t \to \infty} [\, \boldsymbol{r}(t) - \boldsymbol{x}(t)\,] = \boldsymbol{0}$$

- Closed-loop representation



- *Remark*. Offset-free behavior obtain through the design

$$K_r = B^{-R}(A - BK)$$

- PROOF OF THE THEOREM.
  - Since $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ makes the closed-loop asymptotically stable, a final steady state $\boldsymbol{x}_s$ is attained and is characterized by $d\boldsymbol{x}/dt = \boldsymbol{0}$
  - $\begin{cases} \boldsymbol{x}(t) \to \boldsymbol{x}_s \\ \boldsymbol{r}(t) \to \boldsymbol{r}_s \end{cases} \Rightarrow \boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_s - \boldsymbol{B}^{-\mathrm{R}}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{r}_s$
  - Closed-loop system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,[\,-\boldsymbol{K}\,\boldsymbol{x}(t) - \boldsymbol{B}^{-\mathrm{R}}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{r}(t)\,]$$

  - Steady-state relationship

$$\boldsymbol{0} = \boldsymbol{A}\,\boldsymbol{x}_s + \boldsymbol{B}\,[\,-\boldsymbol{K}\boldsymbol{x}_s - \boldsymbol{B}^{-\mathrm{R}}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{r}_s\,]$$

$$\boldsymbol{0} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}_s - \boldsymbol{B}\boldsymbol{B}^{-\mathrm{R}}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{r}_s$$

$$\boldsymbol{0} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}_s - \boldsymbol{I}\,(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{r}_s$$

- Since $(\boldsymbol{A}-\boldsymbol{BK})$ is invertible (because by the stabilizing matrix K ensures that $(\boldsymbol{A}-\boldsymbol{BK})$ has no eigenvalue at zero):

$$(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}_s = (\boldsymbol{A}-\boldsymbol{BK})\,\boldsymbol{r}_s$$

$$\boldsymbol{x}_s = (\boldsymbol{A}-\boldsymbol{BK})^{-1}(\boldsymbol{A}-\boldsymbol{BK})\,\boldsymbol{r}_s$$

$$\boldsymbol{x}_s = \boldsymbol{r}_s$$

- Hence, $\boldsymbol{r}_s - \boldsymbol{x}_s = 0$, and using the definitions of the steady-state values $\boldsymbol{r}_s$ and $\boldsymbol{x}_s$ it follows that

$$\lim_{t \to \infty} [\ \boldsymbol{r}(t) - \boldsymbol{x}(t)\ ] = \boldsymbol{0}$$